8 CHAPTER Probability

AIEEE Syllabus : Probability of an event, addition and multiplication theorems on probability and their applications; Conditional probability; Baye's theorem, Probability distribution of a random variate; Binomial distributions and its properties.

Probability or chance is a measure of uncertainty in any event occurring. Life is full of uncertainties. Probability measures the degree of uncertainty between 0 and 1.

BASIC CONCEPTS

Experiment or trial

An operation which results in some well defined outcomes.

Random Experiment

An experiment whose outcome cannot be predicted with certainty is called a random experiment.

Tossing of a coin is a random experiment.

Throwing a die is a random experiment.

Sample Space

The Set of all possible outcomes of an experiment is called the sample space of that experiment.

It is denoted by S.

When a coin is tossed sample space $S = \{H, T\}$

When a die is thrown, sample space $S = \{1, 2, 3, 4, 5, 6\}$

Event : A subset of the sample space is called an event.

TYPES OF EVENTS

Simple event or elementary event

An event is called a simple event if it is a singleton subset of the sample space S.

For example, getting of three heads in simultaneous throw of 3 coins is a simple event.

Mixed event or compound event

A subset of the sample space S which contains more than one element is called a mixed event.

For example, getting the same number on both die, when cast together is a compound event.

Impossible event :

Represented by empty set ϕ.

For example, getting a number 7 when a die is thrown, is an impossible event.

Sure event or certain event :

The event represented by sample space S itself is a sure event or certain event.

For example, the event of getting one red or black card, when a card is drawn from a well shuffled pack of cards is a sure event.

Equally likely events :

Events are said to be equally likely when one does not happen more often that the other.

For example, the event of drawing a black card and the event of drawing a red card are equally likely, when a card is drawn from a pack of 52 cards.

Mutually exclusive or disjoint events :

A set of events is said to be mutually exclusive, if the happening of one excludes the happening of the others.

Thus events $A_1, A_2, ..., A_n$ are mutually exclusive if and only if $A_i \cap A_j = \phi$ for all $i \neq j$.

For example, when husband and wife both appear for an interview for one post, the selection of husband and the selection of wife are mutually exclusive events.

Exhaustive events (cases) :

A set of events is said to be exhaustive if the performance of the experiment always results in the occurrence of at least one of them.

For example, the events $\{1, 2\}, \{3, 4, 5\}$ and $\{6\}$ are exhaustive event for the sample space $\{1, 2, 3, 4, 5, 6\}$.

If $A_1, A_2, ..., A_n$ are exhaustive events then, $A_1 \cup A_2 \cup .... \cup A_n = S$

ALGEBRA OF EVENTS

Let E and F be two events. Then

(i) E' or $\overline{E}$ or E^C stands for the non-occurrence or negation of E.

(ii) $E \cup F$ stands for the occurrence of at least one of E and F (Also denoted by $E + F$, E or F)

(iii) $E \cap F$ stands for the simultaneous occurrence of E and F. (Also denoted by EF or E and F)

(iv) $E' \cap F'$ stands for the non-occurrence of both E and F

(v) $E \subseteq F$ stands for the occurrence of E implies occurrence of F.

If $E \cap F = \phi \Rightarrow E$ and F are mutually exclusive events.

(vi) $E–F$ denotes the occurrence of event E but not F

$E–F = E \cap F'$

(vii) $E \cup F = F \cup E$ and $E \cap F = F \cap E$

(viii) $(E \cup F)' = E' \cap F'$ and $(E \cap F)' = E' \cup F'$

PROBABILITY

Probability of occurrence of an event is a number lying between 0 and 1 i.e. $0 \le P(E) \le 1$

Let S be the sample space, then the probability of occurrence of an event E is denoted by $P(E)$ and is defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

$$= \frac{\text{number of cases favourable to event } E}{\text{total number of cases}}$$

If $P(E) = 0$ Event is impossible

$P(E) = 1$ Event is sure

Remarks :

1. More is the probability of an event, more are chances of its happening.
2. $P(\phi) = 0$ & $P(S) = 1$

i.e. nothing outside sample space can occur.

ADDITION THEOREM

1. If $p_1, p_2, \ldots\ldots\ldots\, p_n$ be the probabilities of n mutually exclusive events $E_1, E_2, \ldots\ldots\, E_n$, then the probability p, that any one of these events will happen is given by

   $p = p_1 + p_2 + \ldots\ldots\ldots + p_n$

   or $P(E_1 \cup E_2 \cup \ldots\ldots\ldots \cup E_n) = P(E_1) + P(E_2) + \ldots\ldots\ldots + P(E_n)$

2. If E and F are any two events (not mutually exclusive), then

   $P(E \cup F) = P(E + F) = P(E) + P(F) - P(E \cap F) = P(E) + P(F) - P(EF)$

   In general, If $E_1, E_2, \ldots\ldots\ldots\, E_n$ are n events, then

$$P(E_1 \cup E_2 \cup \ldots\ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{1 \le i < j \le n} P(E_i \cap E_j) + \sum_{1 \le i < j < k \le n} P(E_i \cap E_j \cap E_k) - \ldots\ldots\ldots + (-1)^{n-1} P(E_1 \cap E_2 \cap \ldots\ldots\ldots \cap E_n)$$

IMPORTANT RESULTS

(i) If E and F are two events then

P (exactly one of E, F occurs) $= P(E \cup F) - P(E \cap F) = P(E' \cup F') - P(E' \cap F')$

(ii) $P(E' \cup F') = 1 - P(E \cap F)$, $P(E' \cap F') = 1 - P(E \cup F)$

(iii) If $E_1, E_2, \ldots\ldots\ldots\, E_n$ are n events then

$P(E_1 \cup E_2 \cup \ldots\ldots\ldots \cup E_n) \le P(E_1) + P(E_2) + \ldots\ldots\ldots + P(E_n)$

(iv) $P(E_1 \cap E_2 \cap \ldots\ldots\ldots \cap E_n) \ge 1 - P(E'_1) - P(E'_2) - \ldots\ldots\ldots - P(E'_n)$

(v) $P(E_1 \cap E_2 \cap .... \cap E_n) \geq P(E_1) + P(E_2) + ... + P(E_n) - (n-1)$

(vi) If $E \subseteq F \Rightarrow P(E) \leq P(F)$

Conditional Probability :

The probability of occurrence of an event E, given that F has already occurred is called the conditional probability of occurrence of E. It is denoted by $P(E/F)$.

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{n(E \cap F)}{n(F)} \qquad (P(F) \neq 0)$$

$$\Rightarrow P(E/F) = \frac{N_{E \cap F}/N}{N_F/N} = \frac{P(E \cap F)}{P(F)}$$

(N = total no. of outcomes)

Hence, $P(E \cap F) = \begin{cases} P(F) \,.\, P(E/F) & (\text{if } P(F) \neq 0) \\ P(E) \,.\, P(F/E) & (\text{if } P(E) \neq 0) \end{cases}$

Independent Events :

Two events E and F are independent if the occurrence or non-occurrence of $E(F)$ does not affect the probability of occurrence or non-occurrence of $F(E)$, that is

$P(F/E) = P(F)$ provided $P(E) \neq 0$.

Here $P(F/E)$ stands for probability of F if E has already happened. Thus, E and F are independent if and only if $P(E \cap F) = P(E) \,.\, P(F)$

Multiplication Laws of Probability :

If $E_1, E_2, ... E_{n+1}$ be $(n+1)$ events such that $P(E_1 \cap E_2 \cap .... \cap E_n) > 0$ then.

$$P(E_1 \cap E_2 \cap .... \cap E_{n+1}) = P(E_1).\, P(E_2/E_1).P(E_3/E_1 \cap E_2) \,.......\, P(E_{n+1}/E_1 \cap E_2 \cap .... \cap E_n)$$

Special Cases :

(i) If $E_1, E_2 ...... E_n$ are independent events. Then $P(E_1 \cap E_2 .... \cap E_n) = P(E_1)\, P(E_2) .... P(E_n)$

(ii) The probability that none of $E_1, ... E_n$ happen is $= (1 - P(E_1)) \,.\, (1 - P(E_2)) .... (1 - P(E_n))$

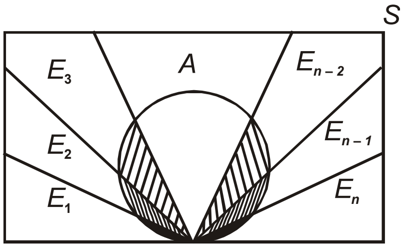
(iii) If p is the probability that an idependent event happens in one trial, then the probability that it will happen in a succession of k trials $= p^k$

Total Probability Rule :

If $\{E_i\}$, $i = 1, 2.......n$ be n events such that $E_i \cap E_j = 0$ for j and $\neq i$ $\bigcup_{i=1}^{n} E_i = S$ (they are mutually exclusive and exhaustive). Suppose $P(E_i) > 0$ $(i = 1, 2.....n)$. Then for any event A,

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i) = P(E_1)\, P(A/E_1) + ... + P(E_n)\, P(A/E_n).$$

Baye's Rule :



Let $\{E_i\}$ be mutually exclusive events such that $P(E_i) > 0$ for $i = 1, \ldots, n$ and $S = \bigcup_{i=1}^{n} E_i$. Let A be any event with $P(A) > 0$. Then for $i = 1, 2, \ldots, n$

$$P(E_i/A) = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(E_i).P(A/E_i)}$$

Binomial Distribution

(i) If n independent trials are performed and if p be the probability of success of each trial and q of its failures, then the probabilities of getting 0, 1, 2, 3,........n successes are given by the respective terms of the Binomial expansion.

$$(q+p)^n = q^n + {}^nC_1\, q^{n-1}\, p + {}^nC_2\, q^{n-2}\, p^2 + \ldots\ldots\ldots + p^n$$
$$= P(0) + P(1) + P(2) + \ldots\ldots\ldots + P(n)$$

The values of p and q remain the same for all trials in an experiment

(ii) The probability of exactly r successes in n trials is given by

$$P(r) = {}^nC_r \,.\, q^{n-r} \,.\, p^r$$

(iii) The probability of at least one success

$= P(1) + P(2) + \ldots\ldots + P(n) = 1 - P(0)$

The probability of at least two successes

$= 1 - [P(0) + P(1)]$ and so on

Note **:** *The mean, variance and standard deviation of a binomial distribution are np, npq and* $\sqrt{npq}$ *respectively.*