8 CHAPTER Inverse Trigonometric Functions

AIEEE Syllabus

Inverse Trigonometric Functions and their Properties

INVERSE FUNCTION

Definition

If a function is one to one and onto from A to B, then function g which associates each element $y \in B$ to one and only one element $x \in A$, such that $y = f(x)$, then g is called the inverse function of f, denoted by $x = g(y)$.

Usually we denote $g = f^{-1}$ {Read as f inverse}

$\therefore \quad x = f^{-1}(y)$.

Inverse Trigonometric Functions

We have seen that the trigonometric functions, sin, cos etc. are all periodic and thus, each of them achieves the same numerical value at an infinite number of points. Thus, the equation $\sin x = \frac{1}{2}$ has an infinite number of solutions, *viz.*, $x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$ etc. If one is to answer the question : "What is the angle whose sine is $\frac{1}{2}$?", there is no unique answer. The difficulty arises as the function $f : R \to R$ defined by $f(x) = \sin x$ is not one to one and thus, does not admit of an inverse. To achieve a unique answer to the aforesaid question, we restrict the domain and codomain of $\sin x$ so that the resulting function is invertible. Thus, the function $g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by $g(x) = \sin x$ is one to one and onto and admits of an inverse (denoted

b y

$h = \sin^{-1}$ and read as sine inverse or arc sin) defined as $h : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $h(y) = x$ if $y = \sin x$. The function $\sin^{-1}$ is the inverse of the sine function when the sine function is viewed in a restricted sense.

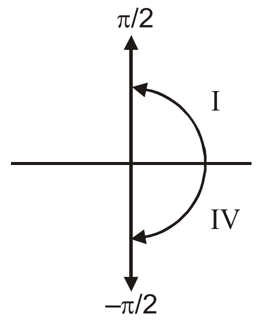
We similarly define the other inverse trigonometric functions

Important Points

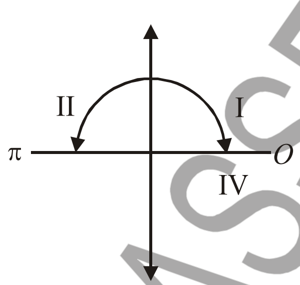
1. $\sin^{-1}x$ is an angle and denotes the smallest numerical angle, whose sine is x.
2. If there are two angles one positive and the other negative having same numerical value. Then we shall take the positive value.

Intervals for Inverse Functions

Here, $\sin^{-1}x$, $\text{cosec}^{-1}x$, $\tan^{-1}x$ belongs to I and IV quadrant.



Here, $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ belongs to I and II quadrant.



1. I quadrant is common to all the inverse functions.
2. III quadrant is not used in inverse function.
3. IV quadrant is used in the clockwise direction *i.e.*, $-\pi/2 \le y \le 0$.

DOMAIN, RANGE AND GRAPHS OF INVERSE FUNCTIONS

1. If $\sin y = x$, then $y = \sin^{-1}x$, under certain condition.

$-1 \le \sin y \le 1$; but $\sin y = x$.

$\therefore \quad -1 \le x \le 1$

Again, $\sin y = -1 \Rightarrow y = -\pi/2$

and $\sin y = 1 \Rightarrow y = \pi/2$

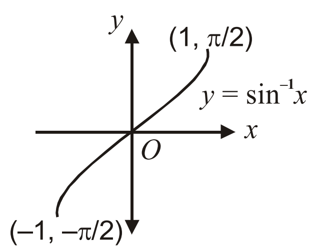
Keeping in mind numerically smallest angles or real numbers.

$\therefore \qquad -\pi/2 \le y \le \pi/2$

These restrictions on the values of x and y provide us with the domain and range for the function $y = \sin^{-1}x$.

i.e., Domain : $x \in [-1, 1]$

Range : $y \in [-\pi/2, \pi/2]$



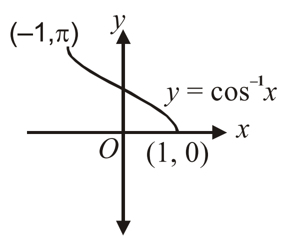
2. Let $\cos y = x$ then $y = \cos^{-1}x$, under certain condition $-1 \le \cos y \le 1$.

$\Rightarrow \quad -1 \le x \le 1$

$\cos y = -1 \Rightarrow y = \pi$

$\cos y = 1 \Rightarrow y = 0$

$\therefore \quad 0 \le y \le \pi$ {as $\cos x$ is a decreasing function in $[0, \pi]$;



hence $\cos\pi \le \cos y \le \cos 0$}

These restrictions on the values of x and y provide us the domain and range for the function $y = \cos^{-1}x$.

i.e., Domain : $x \in [-1, 1]$

Range : $y \in [0, \pi]$

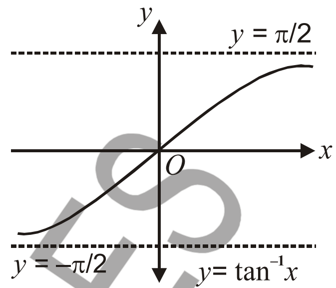
3. If $\tan y = x$ then $y = \tan^{-1}x$, under certain conditions.

Here, $\tan y \in R \Rightarrow x \in R$

$-\infty < \tan y < \infty \Rightarrow -\pi/2 < y < \pi/2$

Thus, domain $x \in R$

Range $y \in (-\pi/2, \pi/2)$



4. If $\cot y = x$, then $y = \cot^{-1}x$ (under certain conditions)

$\cot y \in R \Rightarrow x \in R$;

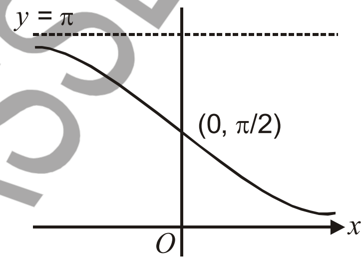
$-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

These conditions on x and y make the function, $\cot y = x$ one-one and onto so that the inverse function exists.

i.e., $y = \cot^{-1}x$ is meaningful.

i.e., Domain : $x \in R$

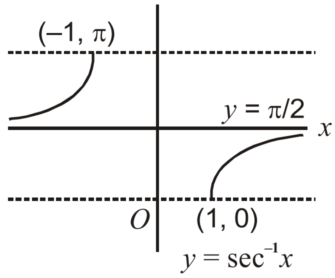
Range : $y \in (0, \pi)$



5. If $\sec y = x$, then $y = \sec^{-1}x$, where $|x| \ge 1$ and $0 \le y \le \pi$, $y \ne \pi/2$

Here, Domain : $x \in R - (-1, 1)$

Range : $y \in [0, \pi] - \{\pi/2\}$

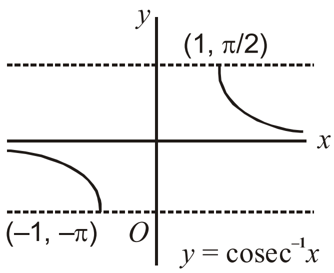


6. If $\operatorname{cosec} y = x$ then $y = \operatorname{cosec}^{-1}x$,

where $|x| \ge 1$ and $-\pi/2 \le y \le \pi/2$, $y \ne 0$

Here, domain $\in R - (-1, 1)$

Range $\in [-\pi/2, \pi/2] - \{0\}$



Function	**Domain**	**Codomain = Range**
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$

$\tan^{-1}$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	R	$(0, \pi)$
$\sec^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
cosec^{-1}	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
Thus, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$; $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ etc.		
$\sin\frac{5\pi}{6} = \frac{1}{2}$ but $\left(\frac{5\pi}{6}\right) \neq \sin^{-1}\frac{1}{2}$		

$\sin^{-1}\left(\frac{1}{2}\right) = \arcsin\frac{1}{2} = \frac{\pi}{6}$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property I

(i) $\sin^{-1}(\sin\theta) = \theta$ only if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos\theta) = \theta$ only if $0 \leq \theta \leq \pi$

(iii) $\tan^{-1}(\tan\theta) = \theta$ only if $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\cot^{-1}(\cot\theta) = \theta$ only if $\theta \in (0, \pi)$

(v) $\sec^{-1}(\sec\theta) = \theta$ only if $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

(vi) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$ only if $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Property II

(i) $\sin\sin^{-1}x = x$: $-1 \leq x \leq 1$

(ii) $\cos\cos^{-1}x = x$: $-1 \leq x \leq 1$

(iii) $\tan\tan^{-1}x = x$: $x \in R$

(iv) $\cot\cot^{-1}x = x$: $x \in R$

(v) $\sec\sec^{-1}x = x$: $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\operatorname{cosec}\, \operatorname{cosec}^{-1} x = x \quad : \quad x \in (-\infty, -1] \cup [1, \infty)$

Property III

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

Property IV

(i) $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$ if $-1 \le x < 0$ or $0 < x \le 1$

(ii) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$ if $-1 \le x < 0$ or $0 < x \le 1$

(iii) $\cot^{-1} x = \tan^{-1}\frac{1}{x}$ if $x > 0$

(iv) $\cot^{-1} x = \pi + \tan^{-1}\left(\frac{1}{x}\right)$ if $x < 0$

(v) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

(vi) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

(vii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, \text{ for } x > 0 \\ -\pi + \cot^{-1} x, \text{ for } x < 0 \end{cases}$

Property V

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad : \quad -1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad : \quad x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$ if $x \le -1$ or $x \ge 1$

Property VI

(i) $$\sin^{-1}x+\sin^{-1}y=\begin{cases}-\pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right] & \text{if } x,y<0, x^2+y^2>1\\ \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) & \text{if } x^2+y^2\le 1 \text{ or } xy<0 \text{ and } x^2+y^2>1\\ \pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right] & \text{if } x,y>0, x^2+y^2>1\end{cases}$$

(ii) $$\sin^{-1}x-\sin^{-1}y=\begin{cases}-\pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\right] & \text{if } x>0, y<0, x^2+y^2>1\\ \sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right) & \text{if } x^2+y^2\le 1 \text{ or } xy>0 \text{ and } x^2+y^2>1\\ \pi-\left[\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)\right] & \text{if } x<0, y>0, x^2+y^2>1\end{cases}$$

(iii) $$\tan^{-1}x+\tan^{-1}y=\begin{cases}\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } x,y>0 \text{ and } xy>1\\ \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } xy<1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if } x,y<0 \text{ and } xy>1\end{cases}$$

(iv) $$\tan^{-1}x-\tan^{-1}y=\begin{cases}\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if } x>0, y<0 \text{ and } xy<-1\\ \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if } xy>-1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if } x<0, y>0 \text{ and } xy<-1\end{cases}$$

There is no need to memorize all the formulae given in different books. To arrive at the answer one must keep in mind the range of inverse trigonometric functions. We, however, list some widely used formulae for easy reference. The formulae for inverse cosine (cotangent) can be obtained from those for inverse sine (tangent).

Remark : If $x_1, x_2, x_3, \ldots, x_n \in R$, then

$$\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{s_1-s_3+s_5\ldots}{1-s_2+s_4-s_6\ldots}\right)$$

where s_k denotes the sum of the product of $x_1, x_2, \ldots, x_n$ taken k at a time.

Property VII

(i) $$\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\text{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

(iii) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Property VIII

(i) $$2\sin^{-1}x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right), \text{ if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), \text{ if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), \text{ if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

(ii) $$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3), \text{ if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3), \text{ if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), \text{ if } -1 \le x < -\frac{1}{2} \end{cases}$$

(iii) $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1); \text{ if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1); \text{ if } -1 \le x \le 0 \end{cases}$$

(iv) $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x), \text{ if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), \text{ if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), \text{ if } -1 \le x \le -\frac{1}{2} \end{cases}$$

(v) $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right); \text{ if } -1 < x \le 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right); \text{ if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right); \text{ if } x < -1 \end{cases}$$

(vi) $$3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); \text{if } -\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); \text{if } x>\dfrac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); \text{if } x<\dfrac{-1}{\sqrt{3}}\end{cases}$$

(vii) $$2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); \text{if } -1\le x\le 1\\ \pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); \text{if } x>1\\ -\pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); \text{if } x<-1\end{cases}$$

(viii) $$2\tan^{-1}x=\begin{cases}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); \text{if } 0\le x<\infty\\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); \text{if } -\infty<x\le 0\end{cases}$$