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CHAPTER

Binomial Theorem and its Application

AIEEE Syllabus : Binomial theorem for a positive integral index, General term and middle term, Properties of Binomial coefficients and Simple applications

BINOMIAL EXPRESSION

An algebraic expression containing only two terms is called a binomial expression. Such as $a + x$, $2x + 5y$, $2x + \frac{1}{y}$, $a + \frac{b}{x}$ etc. all are binomal expressions.

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

If $x, y \in R$, then $\forall$ n $\in N$

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_ny^n$

Here all ${}^nC_r$'s are called binomial coefficient

$$\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$$

$$\boxed{\begin{array}{l}{}^nC_0 = 1 = {}^nC_n \\ {}^nC_1 = n\end{array}}$$

$$\boxed{{}^nC_r = {}^nC_{n-r};\ 1 \le r \le n}$$

Note : *(i) n is the exponent or index of binomial*

(ii) Total number of terms n + 1

(iii) The $(r + 1)^{th}$ term = $T_{r+1} = {}^nC_rx^{n-r}y^r$

(iv) Sum of exponents of x and y in any term = n

(v) Powers of x go on decreasing by 1 and powers of y go on increasing by 1 in subsequent terms

(vi) In any term suffix of binomial coefficient is power of y

(a) *Replacing y with –y in the expansion of $(x + y)^n$, we get*

$(x - y)^n = {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 - ..... + (-1)^n\,{}^nC_n y^n$

(b) *Replacing x by 1 and y with x, we get*

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

(c) $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - .... + (-1)^n\,{}^nC_nx^n$

(d) $(x + y)^n + (x - y)^n = 2[{}^nC_0x^n + {}^nC_2x^{n-2} + ...]$

= 2[Sum of terms at odd places]

last term ${}^nC_ny^n$ if n is even and ${}^nC_{n-1}xy^{n-1}$ if n is odd

(e) $(x + y)^n - (x - y)^n$ = 2[Sum of terms at even places]

last term ${}^nC_{n-1}xy^{n-1}$ if n is even and ${}^nC_ny^n$ if n is odd

General term in the expansion of $(x + y)^n$

The $(r + 1)^{\text{th}}$ term = T_{r+1} is called the general term.

$$T_{r+1} = nC_rx^{n-r}y^r$$

Middle term in the expansion of $(x + y)^n$

Case 1.

$n = 2m\,;\ m \in N$

Then, number of terms in the expansion = $n + 1 = 2m + 1$; Thus middle term is T_{m+1}

i.e. $\left(\frac{n}{2}+1\right)^{\text{th}}$ term

$$T_{m+1} = {}^nC_{\frac{n}{2}}\,x^{\frac{n}{2}},\,y^{\frac{n}{2}}$$

Case 2.

$n = 2m + 1\,;\ m \in N$

Then, number of terms in the expansion is = $n + 1 = 2m + 2$. There are two middle terms, *viz.* T_{m+1} and T_{m+2}.

i.e. $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ term

$$T_{m+1} = {}^nC_{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}} \cdot y^{\frac{n-1}{2}}$$

$$T_{m+2} = {}^nC_{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}} \cdot y^{\frac{n+1}{2}}$$

p^{th} term from the end in the expansion of $(x + y)^n$

p^{th} term from end = $(n + 2 - p)^{th}$ term from begining

$$= {}^nC_{n-p+1} x^{p-1} y^{(n-p+1)}$$

Numerically greatest term in $(1 + x)^n$

The method for finding the greatest term is as below :

1. Find the value of $K = \dfrac{(n+1)|x|}{1+|x|}$.
2. If K is an integer, then T_K and T_{K+1} both are equal and greatest.
3. If K is not an integer then $T_{[K]+1}$ is the greatest term where $[K]$ is the greatest integral part of K.
4. To find the greatest term in $(x + y)^n = x^n\left(1+\dfrac{y}{x}\right)^n$, find the greatest term in $\left(1+\dfrac{y}{x}\right)^n$ and multiply with x^n.

Properties of Binomial Coefficients

In the expansion of $(1 + x)^n$, the coefficients, ${}^nC_0$, ${}^nC_1$, ${}^nC_2$ ..... ${}^nC_n$ are denoted by C_0, C_1, C_2 .... C_n then.

1. If n is even, then greatest coefficient is ${}^nC_{n/2}$.
2. If n is odd, then greatest coefficients are ${}^nC_{(n-1)/2}$ or ${}^nC_{(n+1)/2}$
3. $r.{}^nC_r = n.{}^{n-1}C_{r-1}$ $0 < r \le n$.
4. $\dfrac{{}^nC_r}{r+1} = \dfrac{{}^{n+1}C_{r+1}}{n+1}$
5. ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{(r-1)}$
6. $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$
7. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$
8. ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$
9. ${}^nC_r = {}^nC_{n-r}$

Note : *(1) The number of terms in the expansion of* $(x + y + z)^n$ *is* $\dfrac{(n+1)(n+2)}{2}$ *where* $n \in N$.

(2) Sum of all coefficients in the expansion of $(x_1 + x_2 + ... + x_K)^n$ *is obtained by keeping each* $x_i = 1$ *so it is* k^n.

Some Relation involving Binomial Coefficients

$(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$ …(1)

Relation (1) is important. Many properties of binomial coefficients are obtained by

(i) Giving different values to x in (1) and/or

(ii) Differentiating (1) with respect to x and then putting convenient values of x, and/or

(iii) Integrating (1) with respect to x and then putting convenient values of x.

Some important relation are given below :

1. $\sum_{r=0}^{n} {}^nC_r = C_0 + C_1 + C_2 + ... C_n = 2^n$

2. $C_0 + C_2 + C_4 .... = C_1 + C_3 + ... = 2^{n-1}$

3. $\sum_{r=1 \text{ or } 0}^{n} r.{}^nC_r = C_1 + 2C_2 + 3C_3 .... + {}^nC_n = n.2^{n-1}$

4. $\sum_{r=0}^{n} \frac{{}^nC_r}{r+1} = \frac{2^{n+1}-1}{n+1}$

5. $C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n = 0$

6. $C_0^2 + C_1^2 + ...C_n^2 = \sum_{r=0}^{n} ({}^nC_r)^2 = {}^{2n}C_n = \frac{2n!}{(n!)^2}$

7. $C_0^2 - C_1^2 + C_2^2 - C_3^2 + (-1)^n C_n^2 .... = \begin{cases} 0, \text{ if } n \text{ is odd} \\ (-1)^{n/2}.{}^nC_{n/2} \text{ if } n \text{ is even} \end{cases}$

8. $C_0C_1 + C_1C_2 + ... + C_{n-1}C_n = {}^{2n}C_{n-1}$

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