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CHAPTER

Permutations and Combinations

AIEEE Syllabus : Fundamental principle of counting; Permutation as an arrangement and combination as selection, Meaning of P(n,r) and C(n,r). Simple applications.

Factorial-notation

$\lfloor n$ or $n!$ is read as 'n factorial'

$\lfloor n = n! \quad = 1.2.3\ldots\ldots (n-1)\,n.$

$m! = n!$ when $m = n$

$1! = 1; \lfloor 0 = 1$

Note : Factorial of negative numbers or proper fractions is not defined.

FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication theorem

If an operation can be performed in 'm' ways and another operation can be performed in 'n' ways, then the two operations in succession can be performed in '$m \times n$' ways. For example: A person can go by rail from Delhi to Lucknow in two ways and from Lucknow to Varanasi in three ways then he can travel from Delhi to Varanasi via Lucknow in $2 \times 3 = 6$ ways

Addition Theorem

If an operation can be performed in 'm' ways and another independent operation can be performed in 'n' ways, then either of the two operations can be performed in '$m + n$' ways.

PERMUTATION AS AN ARRANGEMENT

Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.

Let n and r be non-negative integers such that $0 \le r \le n$. Then

1. **Number of permutation of n different things, taken r at a time, is denoted by ^nP_r or $P(n, r)$.**

$$^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2).......(n-r+1)$$

PERMUTATIONS UNDER RESTRICTIONS

1. Number of permutations of n different things taken r at a time, when one particular thing is to be always included is, $r.^{n-1}P_{r-1}$.
2. Number of permutation of n different things taken r at a time, when p particular things are to be always included, is $\frac{r!}{(r-p)!}.^{n-p}P_{r-p}$

3. Number of permutations of n different things taken r at a time, when p particular things are always together is

$$p!.[r-(p-1)].^{n-p}P_{r-p}$$

4. Number of permutations of n different things, taken r at a time, when p particular things are not to be taken is $^{n-p}P_r$.

5. Number of permutations of n different things, taken all at a time, when p particular things always occur together is

$$p!(n-p+1)!$$

6. Number of permutations of n different things, taken all at a time, when m particular things never occur together is $n! - [m! \, . \, (n-m+1)!]$

CIRCULAR PERMUTATIONS

(1) Arrangements round a circular table

When clockwise and anticlockwise arrangements are considered different then number of circular permutations of n different things taken all at a time is $(n-1)!$

(2) Arrangement of flowers in a garland

When clockwise and anticlockwise arrangements are considered same, then required permutations =

$\frac{1}{2}(n-1)!$

(3) Number of circular permutations of *n* different things taken

r at a time $= (^nP_r) / r$ When clockwise is different than anticlockwise arrangement

$= (^nP_r) / (2r)$ When clockwise & anticlockwise are considered same.

COMBINATION AS SELECTION

Combination is the selection of a group which can be made by some or all of number of things without reference to the order.

Hence just the selections is combination where as selection cum arrangement is permutation.

Generally, number of permutations is larger than number of selections.

The number of combinations of n different things taken r at a time is given by ^nC_r or $C(n, r)$ or $\binom{n}{r}$.

KEY RESULTS ON ^nC_r

(i) $^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}$ (if $r > n$, then $^nC_r = 0$)

(ii) $^nC_r \in N$

(iii) $^nC_0 = {}^nC_n = 1$

(iv) $^nC_1 = n$

(v) $^nC_r = {^nC_{n-r}}$

(vi) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(vii) $n.{^{n-1}C_{r-1}} = (n-r+1).{^nC_{r-1}}$

(viii) greatest value of $^nC_r = {^nC_{n/2}}$ when n is even

$= {^nC_{\frac{n+1}{2}}}$ or $^nC_{\frac{n-1}{2}}$ when n is odd

(ix) $^nC_r = \frac{n}{r}\,{^{n-1}C_{r-1}}$

(x) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(xi) $C_0 + C_1 .... + C_n = 2^n$

(xii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + ... = 2^{n-1}$

(xiii) $^{2n+1}C_0 + {^{2n+1}C_1} + ... + {^{2n+1}C_n} = 2^{2n}$

(xiv) $^nC_n + {^{n+1}C_n} + {^{n+2}C_n} + ... + {^{2n-1}C_n} = {^{2n}C_{n+1}}$

Note : *Product of r consecutive positive integers is divisible by r!.*

SIMPLE APPLICATIONS OF COMBINATION

1. The number of combinations of n different things taken r at a time
   (a) When k particular things always occur is $^{n-k}C_{r-k}$.
   (b) When k particular things never occur is $^{n-k}C_r$.
2. Number of combination of n different things selecting atleast one of them is
   $^nC_1 + {^nC_2} + ...{^nC_n} = 2^n - 1$
3. Number of combination of n identical things, taking r ($r \le n$) at a time is 1.
4. Number of ways of selecting none, one or more things out of n alike things is $(n + 1)$
   (*i.e.* 0, 1, 2,................ or n things)
5. Out of $(p + q + r + s)$ things. p are alike of one kind, q are alike of second kind, r are alike of third kind and s are different each, then total number of combination is
   $(p + 1)(q + 1)(r + 1)(2)^s - 1$
6. Divisors of N

   Let $N = P_1^{\alpha_1}.P_2^{\alpha_2}.P_3^{\alpha_3}.............P_k^{\alpha_k}$ where P_1, P_2.......... P_k are different primes and α_1, α_2.......α_k are natural numbers; then

   (a) Total number of divisors of N including 1 & N is
   $(\alpha_1 + 1)(\alpha_2 + 1).........(\alpha_k + 1)$

(b) Sum of these divisors excluding 1 & N is

$$= (P_1^0 + P_1^1 + ...P_1^{\alpha_1})\,(P_2^0 + P_2^1 + ...P_2^{\alpha_2})..........\,(P_k^0 + P_k^1 + ...P_k^{\alpha_k})$$

(c) The number of ways in which N can be resolved into two factors which are coprime to each other is 2^{n-1} where n is the number of different factors in N.

7. Division into groups

(a) The number of ways in which $(m + n)$ different things can be divided into two groups containing m and n things $(m \neq n)$ respectively are

$$^{m+n}C_m \cdot {}^nC_n = \frac{(m+n)!}{m!n!}$$

(b) If $m = n$ and group order is not important then $\frac{2n!}{(2)!(n!)^2}$

(c) If $m = n$ and groups are distinct then $\frac{2n!}{(n!)^2}$

Generalisation

The number of way in which mn different things can be divided equally among m groups of n each.

(a) If order of group is not important $\frac{(mn)!}{m!(n!)^m}$

(b) If order of group is important $= \frac{(mn)!}{(n!)^m}$

8. Arrangement into groups

(a) To arrange n different things into r different groups.

If blank groups are allowed, then $^{n-r+1}P_n$

If blank groups are not allowed, then $n!.{}^{n-1}C_{r-1}$

(b) To distribute n different things into r different groups

$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n............ (-1)^{r-1}\,{}^rC_{-r}$.

Note : Coefficient of x^r in $e^{px} = \frac{p^r}{r!}$

(c) To distribute n identical things into r different groups

If blank groups are allowed, then $^{n+r-1}C_{r-1}$

If blank groups are not allowed, then $^{n-1}C_{r-1}$

(d) To distribute n identical things in r groups so that no group contains less than l and more than m things $(m > l)$ is coefficient of x^n in the expansion of $(x^l + x^{l+1}...... + x^m)^r$

(e) To select r things from a group of n things having p things identical is

$$\sum_{r=0}^{r} {}^{n-p}C_r \quad \text{if } r \le p$$

and $$\sum_{r=r-p}^{r} {}^{n-p}C_r \quad \text{if } \; r > p$$

9. Derangements

If n things are arranged in a row, the number of ways in which they can be deranged so that no one of them occupies its original place or no object goes to its scheduled place, is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right)$$

Remark :

If r things go to wrong place out of n things, then $(n - r)$ things go to original place (Here $r < n$) D_n = No. of ways, if all n things go to wrong place and D_r = No of ways, if r things goes to wrong place. If r goes to wrong place out of n, then $(n - r)$ goes to correct places.

Then $D_n = {}^nC_{n-r}D_r$.

Where $D_r = r!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^r\frac{1}{r!}\right)$