6 CHAPTER Trigonometric Functions

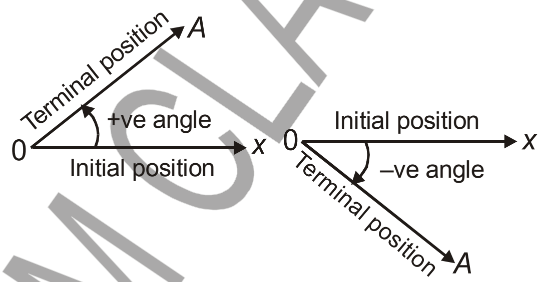
AIEEE Syllabus

Trigonometric functions their periodicity and Graphs, Addition and subtraction formulae, formulae involving multiple and sub-multiple angles.

ANGLES AND THEIR MEASURES

Angle

A figure traced by rotating a given ray about its end point. The measure of angle is the amount of rotation performed from the initial side to the terminal side. Angle performed by anticlockwise rotation are taken as positive whereas angles formed by clockwise rotation are considered as negative.



Radian or Circular Measure

The angle subtended at the centre of a circle by an arc of the same circle whose length is equal to the radius of the circle is called 1 radian and is denoted by 1^c.

When no unit is mentioned with an angle, it is always understood to be in radian.

Radian measure and real numbers are same.

The ratio of circumference and diameter of a circle is always constant and denoted by Greek letter 'π'.

π is an irrational number, $\pi = \dfrac{\text{Circumference of Circle}}{\text{Diameter of circle}}$

Circumference $= 2\pi r = \pi \times$ diameter

$\pi = \dfrac{22}{7}$ (approx) $= 3.1415........$

Arc-angle relation

Angle $= \dfrac{\text{arc}}{\text{radius}}$; Here angle is always in radian.

$\theta = \dfrac{l}{r}$

Angle subtended by a very small arc is approximately calculated by

$\theta = \dfrac{\text{Chord } AB}{r}$

$\therefore$ Arc AB $\approx$ chord AB for small angle θ

Relation between degree and radian.

$$\pi^c = 180^o$$

$$1^c = \left(\frac{180}{\pi}\right)^o$$

$$1^o = \left(\frac{\pi}{180}\right)^c$$

Relation between the sides and interior angles of a polygon

(a) Sum of interior angle of polygon of n sides = $(2n - 4) \times 90° = (n - 2)\pi^c$

(b) Each interior angle of a regular polygon of n sides $= \dfrac{(2n-4)90°}{n} = \dfrac{(n-2)}{n}\pi^c$

TRIGONOMETRIC FUNCTIONS (T-RATIOS)

Trigonometric Functions

(i) $\sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{MP}{OP}$

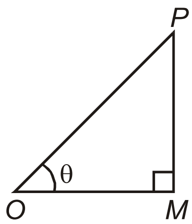
(ii) $\cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{OM}{OP}$

(iii) $\tan\theta = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{MP}{OM}$

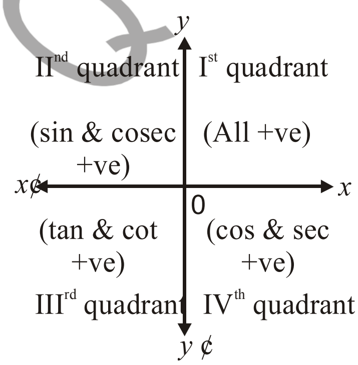
(iv) $\cot\theta = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{OM}{MP}$

(v) $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{OP}{OM}$

(vi) $\text{cosec}\,\theta = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{OP}{MP}$



Signs of T-Ratios



Domain and Range of Trigonometric Functions

Function	Domain	Range
$\sin\theta$	R	$[-1, 1]$
$\cos\theta$	R	$[-1, 1]$
$\tan\theta$	$R \sim \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	R
$\cot\theta$	$R \sim \{n\pi : n \in I\}$	R
$\sec\theta$	$R \sim \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	$(-\infty, -1] \cup [1, \infty)$
$\text{cosec}\,\theta$	$R \sim \{n\pi : n \in I\}$	$(-\infty, -1] \cup [1, \infty)$

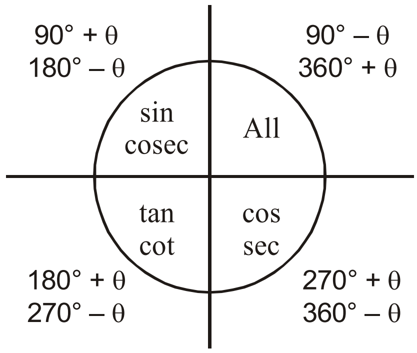
Allied Angle

If θ is any angle then, $-\theta$, $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$ etc. are called as allied angles of θ.

1. To find the sign (+ or –)

   Use the original ratio to find '+ or –' sign (to be affixed) making use of the quadrant rule.

   Thus, ratios shown inside the circle are positive in the corresponding quadrant while other ratios are negative there



2. To find the final ratio

   (a) If π, 2π etc. are present, then there is no change ; i.e., sin remains sin ; cos remains cos etc.

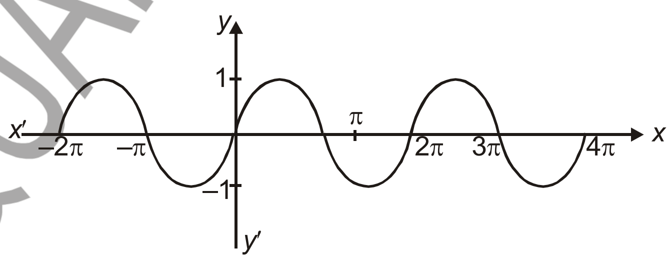
   (b) If $\frac{\pi}{2}, \frac{3\pi}{2}$ are present, then, there is a change as given below :

   sin $\rightleftharpoons$ cos   tan $\rightleftharpoons$ cot   cosec $\rightleftharpoons$ sec

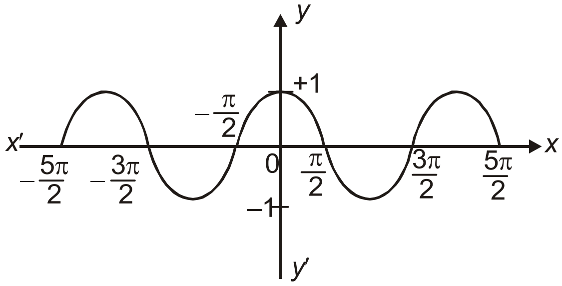
	$-\theta$	$90° - \theta$	$90° + \theta$	$180° - \theta$	$180° + \theta$	$270° - \theta$	$270° + \theta$	$360° - \theta$	$360° + \theta$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

Graphs of standard T-Functions

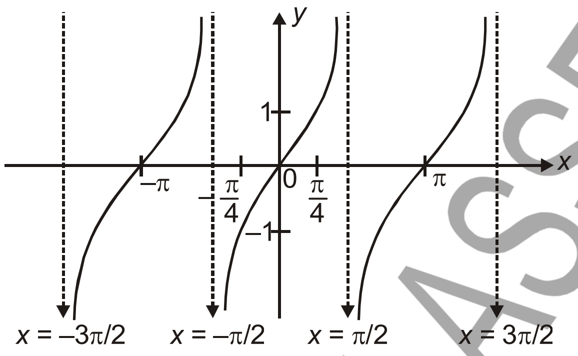
$y = \sin x$



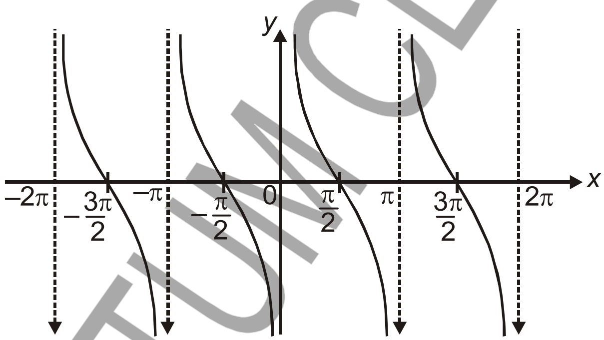
$y = \cos x$



$y = \tan x$



$y = \cot x$



TRIGONOMETRIC IDENTITIES

Fundamental Identities

(i) $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta \Rightarrow \cos^2\theta = 1 - \sin^2\theta$

(ii) $1 + \tan^2\theta = \sec^2\theta \Rightarrow \sec^2\theta - \tan^2\theta = 1$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta \Rightarrow \text{cosec}^2\theta - \cot^2\theta = 1$

Also note the range within which different trigonometric function lie

(1) $-1 \le \sin\theta \le 1$; $|\sin\theta| \le 1$

(2) $-1 \le \cos\theta \le 1$; $|\cos\theta| \le 1$

(3) $0 \le \sin^2\theta \le 1$; $0 \le \cos^2\theta \le 1$

(4) $\text{cosec}\,\theta \le -1$ or $\text{cosec}\,\theta \ge 1$

(5) $\sec\theta \le -1$ or $\sec\theta \ge 1$

(6) $0 < \cos A < \dfrac{\sin A}{A} < \dfrac{1}{\cos A}$; $0 < A < \dfrac{\pi}{2}$

(7) If θ <<< then $\sin\theta \simeq \theta$

Note : *Each trigonometric ratio can be expressed in terms of all other t-ratios e.g.*

$$\sin\theta = \frac{\pm 1}{\sqrt{1+\cot^2\theta}};\ \cos\theta = \frac{\pm\cot\theta}{\sqrt{1+\cot^2\theta}}$$

$$\tan\theta = \frac{1}{\cot\theta};\ \sec\theta = \frac{\pm\sqrt{1+\cot^2\theta}}{\cot\theta}$$

$$\text{cosec}\,\theta = \pm\sqrt{1+\cot^2\theta}$$

Addition and Subtraction Formulae (Compound Angle)

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
3. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
4. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
5. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
7. $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$
8. $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
9. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
10. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
11. $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

    or

    $= \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$
12. $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

    or

    $= \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$
13. $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

14. $\tan\left(\frac{\pi}{4}+A\right)=\frac{1+\tan A}{1-\tan A}$

15. $\tan\left(\frac{\pi}{4}-A\right)=\frac{1-\tan A}{1+\tan A}$

Two special series :

1. $\sin(\alpha)+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots\ldots \sin(\alpha+(n-1)\beta)$

$$=\frac{\sin\left[\alpha+(n-1)\left(\frac{\beta}{2}\right)\right].\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

2. $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots+\cos(\alpha+(n-1)\beta)$

$$=\frac{\cos\left[\alpha+(n-1)\left(\frac{\beta}{2}\right)\right].\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Note : *Note that the angles are in A.P.*

TRANSFORMATION FORMULAE

Product into sum and difference

1. $2\sin A\cos B=\sin(A+B)+\sin(A-B),\ A>B$
2. $2\cos A\sin B=\sin(A+B)-\sin(A-B),\ A>B$
3. $2\cos A\cos B=\cos(A+B)+\cos(A-B)$
4. $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Sum and Difference into products

1. $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

2. $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

3. $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

4. $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

5. $\tan C+\tan D=\frac{\sin(C+D)}{\cos C\cos D}$

6. $\tan C - \tan D = \dfrac{\sin(C - D)}{\cos C \cos D}$

7. $\cot C + \cot D = \dfrac{\sin(C + D)}{\sin C \sin D}$

8. $\cot C - \cot D = \dfrac{\sin(D - C)}{\sin C \sin D}$

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES

T-Ratios of multiple angles : (An angle of the form $n\theta$, $n \in I$)

1. $\sin 2A = 2 \sin A \cos A = \dfrac{2\tan A}{1 + \tan^2 A}$

2. $\cos 2A = \cos^2 A - \sin^2 A$

   $= 2\cos^2 A - 1$

   $= 1 - 2\sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

   Thus,

   $1 + \cos 2A = 2\cos^2 A$

   $1 - \cos 2A = 2\sin^2 A$

3. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$, $\quad \cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

4. (i) $\sin 3A = 3\sin A - 4\sin^3 A$

   (ii) $\cos 3A = 4\cos^3 A - 3\cos A$

   (iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$, $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

5. $\cos A \cos 2A \cos 2^2 A \,........\cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

T-Ratios of submultiple angle

$\left(\text{An angle of the from } \dfrac{\theta}{n}, n \in I\right)$

1. $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \dfrac{2\tan\theta/2}{1 + \tan^2\theta/2}$

2. $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$

$= \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

3. $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$

4. $\cot\theta = \dfrac{\cot^2\frac{\theta}{2} - 1}{2\cot\theta/2}$

5. $\cos^2\frac{\theta}{2} = \dfrac{1+\cos\theta}{2}$

6. $\sin^2\frac{\theta}{2} = \dfrac{1-\cos\theta}{2}$

7. $\tan^2\frac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

8. $\cot^2\frac{\theta}{2} = \dfrac{1+\cos\theta}{1-\cos\theta}$

9. $\dfrac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$

10. $\dfrac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$

T-Ratio of some special angles

(i) $\sin 15° = \cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15° = \sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15° = \cot 75° = 2-\sqrt{3}$

(iv) $\cot 15° = \tan 75° = 2+\sqrt{3}$

(v) $\sin 18° = \cos 72° = \dfrac{\sqrt{5}-1}{4}$

(vi) $\cos 18° = \sin 72° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(vii) $\sin 36° = \cos 54° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

(viii) $\cos 36° = \sin 54° = \dfrac{\sqrt{5}+1}{4}$

(ix) $\tan 22\tfrac{1}{2}° = \sqrt{2}-1$

(x) $\cot 22\tfrac{1}{2}° = \sqrt{2}+1$

Remember :

(i) $\left|\sin\frac{A}{2} + \cos\frac{A}{2}\right| = \sqrt{1+\sin A}$

or $\quad \sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}$ *i.e.*, $\begin{cases} +, \text{ if } 2n\pi - \frac{\pi}{4} \le \frac{A}{2} \le 2n\pi + \frac{3\pi}{4} \\ -, \text{ otherwise} \end{cases}$

(ii) $\left|\sin\frac{A}{2} - \cos\frac{A}{2}\right| = \sqrt{1-\sin A}$

or $\quad \left(\sin\frac{A}{2} - \cos\frac{A}{2}\right) = \pm\sqrt{1-\sin A}$; *i.e.*, $\begin{cases} +, \text{ if } 2n\pi - \frac{\pi}{4} \le \frac{A}{2} \le 2n\pi + \frac{5\pi}{4} \\ -, \text{ otherwise} \end{cases}$

GREATEST AND LEAST VALUES OF $a\cos\theta + b\sin\theta$

$S = a\cos\theta + b\sin\theta$

$= r\left(\frac{a}{r}\cos\theta + \frac{b}{r}\sin\theta\right) \quad : \quad r = \sqrt{a^2+b^2}$

$= r\left(\sin(\theta+\alpha)\right) \quad ; \ \sin\alpha = \frac{a}{r} \quad ; \quad \cos\alpha = \frac{b}{r}$

Since $-1 \le \sin(\theta+\alpha) \le 1$, therefore, $-r \le S \le r$

CONDITIONAL IDENTITIES

When, three angles *A, B, C* satisfy some given relation, several identities can be established connecting the trigonometric ratios of these angles

In a triangle *ABC*, $A + B + C = \pi$;

$\therefore \quad \sin(A+B) = \sin(\pi - C) = \sin C$

and $\quad \cos(A+B) = \cos(\pi - C) = -\cos C$

Also, $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$; Hence,

$$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2}-\frac{C}{2}\right) = \cos\frac{C}{2}$$

$$\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi}{2}-\frac{C}{2}\right) = \sin\frac{C}{2}$$

Remember :

If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iv) $\sin A + \sin B + \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(vii) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(viii) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$