5
CHAPTER

Sequences and Series

AIEEE Syllabus : Arithmetic and Geometric progressions. Insertion of Arithmetic, Geometric means between two given numbers. Relation between A.M. and G.M. Sum upto n terms of Special Series : Sn, Sn2, Sn3 . Arithmetico-Geometric progression.

SEQUENCE, PROGRESSION AND SERIES

A sucession of numbers t_1, t_2, ... t_n formed according to the some definite rule is called sequence.

"A sequence is a function of natural numbers with codomain as the set of Real numbers or complex numbers"

Domain of sequence	$= N$
if Range of sequence	$\subseteq R \Rightarrow$ Real sequence
if Range of sequence	$\subseteq C \Rightarrow$ Complex sequence

Sequence is called finite or infinite depending upon its having number of terms as finite or infinite respectively.

For example: 2, 3, 5, 7, 11, ... is a sequence of prime numbers. It is an infinite sequence.

A progression is a sequence having its terms in a definite pattern e.g.: 1, 4, 9, 16, ... is a progression as each successive term is obtained by squaring the next natural number.

However a sequence may not always have an explicit formula of n^{th} term.

Series is constructed by adding or subtracting the terms of a sequence e.g., 2 + 4 + 6 + 8 + ... is **a series**. The term at n^{th} place is denoted by T_n and is called general term of a sequence or progression or **series**.

ARITHMETIC PROGRESSION (A.P.)

It is sequence in which the difference between any term and its just preceding term remains constant throughout. This constant is called the "common difference" of the A.P. and is denoted by 'd' generally.

A.P. is of the form

$a, (a + d), (a + 2d)$....

where 'a' denotes the first term or initial term

Important relations :

$a_n - a_{n-1} = d$ = common difference $a_n = n^{th}$ term of A.P. = $\{a + (n-1)d\} = l$
a'_r = r^{th} term of A.P. from the end = $(n - r + 1)^{th}$ term from beginning n = total number of terms *i.e.*, $a'_r = a_{(n-r+1)} = a + (n-r)d$
a'_n = n^{th} term of A.P. from the end = $\{l - (n-1)d\}$
S_n = the sum of first n terms of A.P. $= \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$ $= \frac{n}{2}[2l-(n-1)d]$ $a_n = S_n - S_{n-1}$

Properties of Arithmetic Progressions

1. If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., then
   (a) $ka_1 + k, a_2, \ldots, ka_n + k$ are also in A.P.
   (b) $ka_1, ka_2, \ldots, ka_n$ are also in A.P.
   (c) $\frac{a_1}{k}, \frac{a_2}{k}, \ldots, \frac{a_n}{k}, k \neq 0$ are also in A.P.
2. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two A.Ps., then
   (a) $ka_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$, are also in A.P.
   (b) $ka_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$, are also in A.P.
3. If $a_1, a_2, a_3, \ldots a_n$, are in A.P., then
   (a) $ka_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots = 2a_1 + (n-1)d$
   (b) $a_r = \frac{a_{r-k} + a_{r+k}}{2}, 0 \le k \le n - r$
4. If n^{th} term of a sequence is a linear expression in n then the sequence is an A.P.
5. If the sum of first n terms of a sequence is a quadratic expression in n, then the sequence is an A.P.
6. Three numbers a, b, c are in A.P. if and only if $b - a = c - b$, i.e., if and only if $a + c = 2b$.
7. Any three numbers in an A.P. can be taken as $a - d, a, a + d$. Any four numbers in an A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$. Similarly 5 numbers in A.P. can be taken as $a - 2d, a - d, a, a + d, a + 2d$.

GEOMETRIC PROGRESSION (G.P.)

A sequence (finite or infinite) of non-zero numbers in which every term, except the first one, bears a constant ratio with its preceding term, is called a geometric progression.

The constant ratio, also called the common ratio of the G.P. is usually denoted by r.

For example, in the sequence, 1, 2, 4, 8, ...

$$\frac{2}{1}=\frac{4}{2}=\frac{8}{4}=....=2\text{, which is a constant.}$$

Thus, the sequence is a G.P. whose first term is 1 and the common ratio is 2.

Important Relations

For a G.P.; $a, ar, ar^2, \ldots ar^{n-1}$

$a_n = n^{th}$ term of G.P.$= ar^{n-1} = l$ (last term) where a = first term $\neq 0$ $r = \dfrac{a_n}{a_{n-1}}$ $(r \neq 0)$ $a'_n = n^{th}$ term from end $= \dfrac{l}{r^{n-1}}$ $a'_r = r^{th}$ term from end of a G.P. having n terms $= a_{(n-r+1)}$ term from beginning. $= ar^{(n-r)}$ S_n = Sum of n terms from beginning $= \dfrac{a(r^n-1)}{r-1} = \dfrac{lr-a}{r-1}$ when $r \neq 1$ $= na$ when $r = 1$
S_∞ = Sum of infinite G.P. $= \dfrac{a}{1-r}$; where $\lvert r \rvert < 1$

Properties of Geometrical Progression

(i) $a_1, a_2, a_3, \ldots.$ are in G.P. then
$a_1k, a_2k, a_3k, \ldots$ and $a_1/k, a_2/k, a_3/k \ldots.$ are also in G.P. $(k \neq 0)$

(ii) If $a_1, a_2, a_3, \ldots.$ are in G.P. Then $1/a_1, 1/a_2, 1/a_3, \ldots$ are also in G.P.

(iii) If $a_1, a_2, a_3, \ldots.$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $a_1/b_1, a_2/b_2, a_3/b_3, \ldots$ are also in G.P.

(iv) If $a_1, a_2, a_3, \ldots.$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are not in G.P.

(v) If $a_1, a_2, a_3, \ldots.$ are in G.P. $(a_i > 0, \forall i)$, then $\log a_1, \log a_2, \log a_3, \ldots.$ are in A.P. In this case the converse also holds good.

(vi) If $a_1, a_2, a_3, \ldots.a_n$ are in G.P. , then
(a) $a_1a_n = a_2a_{n-1} = a_3a_{n-2} = \ldots. = a^2 r^{n-1}$
(b) $a_r = \sqrt{a_{r-k}a_{r+k}}, \quad 0 \le k \le n-r$

(vii) If $a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n$ are in G.P.

then $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_n}{a_{n-1}}$

$\Rightarrow a_2^2 = a_3 a_1, a_3^2 = a_2 a_4, \ldots$

also $a_2 = a_1 r,\ a_3 = a_1 r^2,\ a_4 = a_1 r^3, \ldots, a_n = a_1 r^{n-1}$

where r is the common ratio.

(viii) Three numbers in G.P. can be taken as $\frac{a}{r}, a, ar$; Five numbers in G.P. can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ etc.

In general: $(2m + 1)$ numbers in G.P. can be written as $(m \in N)$

$$\frac{a}{r^m}, \frac{a}{r^{m-1}}, \ldots, \frac{a}{r}, a\ ar, \ldots, ar^{m-1}, ar^m$$

(ix) Four numbers in G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$; Six numbers in G.P. can be taken as $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$; etc.

In general: $(2m)$ numbers in G.P. can be written as $(m \in N)$

$$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, \ldots, \frac{a}{r^3}, \frac{a}{r}\ ar, ar^3, \ldots, ar^{2m-3}, ar^{2m-1}$$

RECOGNIZATION OF AP & GP

If a, b, c are three successive terms of a sequence

If $\frac{a-b}{b-c} = \frac{a}{a} = 1$, then a, b, c are in A.P.

If $\frac{a-b}{b-c} = \frac{a}{b}$, then a, b, c are in G.P.

MEANS

Arithmetic Mean

If three terms 'a, b, c' are in A.P., then middle term 'b' is called A.M. between the other two i.e., a and c

$$b = \frac{a+c}{2}$$

i.e., A.M. of two numbers x_1 and x_2 is $\frac{x_1 + x_2}{2}$

A.M. of n positive numbers $= \frac{a_1 + a_2 + a_3 \ldots + a_n}{n}$

Insertion of *n* Arithmetic means between two numbers

Let $A_1, A_2, \ldots A_n$ are n A.M. between a and b then

$a, A_1, A_2, \ldots, A_n, b$ form an A.P.

b is $(n + 2)^{\text{th}}$ term

$\therefore\ b = a + (n+1)d \Rightarrow d = \frac{(b-a)}{n+1}$

$$\boxed{A_1 + A_2 + ... + A_n = \frac{n}{2}(a+b)}$$

$$A_r = a + \frac{r(b-a)}{n+1}$$

Geometric Mean

If three terms a, b, c are in G.P., then b is called G.M. of a and c such that

$$\boxed{b = \sqrt{ac}}$$

G.M. of n numbers = $\sqrt[n]{a_1.a_2.a_3....a_n}$

Insertion of *n* G.M. between two numbers (*a* and *b*)

Here a, G_1, G_2, ... , G_n, b will be in G.P.

So $b = (n+2)^{\text{th}}$ term of G.P.

Hence, $b = a \,.\, r^{n+1} \quad \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$$G_k = ar^k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$$

Relations between A.M. and G.M.

For two real positive numbers a and b

$$A = \frac{a+b}{2},\ G = \sqrt{ab}$$

$A > G \quad$ if $a \neq b$ ...(i)

$A = G \quad$ if $a = b$ ...(ii)

So combining (i) & (ii), we get

A ≥ G, the equality holds when $a = b$

If a_1, a_2, a_3... a_n are n positive numbers, then

Above discussion leads to the result that, $\dfrac{a_1 + a_2 + a_3 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 .... a_n}$

Special Series

Sigma (Σ) notation

Σ indicates sum i.e., $\sum_{i=1}^{n} i = \sum n = 1 + 2 + 3 + ... + n$

(i) $\sum_{i=1}^{n} \frac{i+1}{i+2} = \frac{1+1}{1+2} + \frac{2+1}{2+2} + \frac{3+1}{3+2} + ... + \frac{n+1}{n+2}$

(ii) $\sum_{i=1}^{m} a = a + a + ... + a \;\; m$ times

$= am$ where a is constant

(iii) $\sum_{i=1}^{m} ai = a\sum_{i=1}^{m} i = a(1 + 2 + ... + m)$

(iv) $\sum_{i=1}^{m} (i^3 - 2i^2 + i) = \sum_{i=1}^{m} i^3 - 2\sum_{i=1}^{m} i^2 + \sum_{i=1}^{m} i$

Important Results

(i) Sum of first n natural numbers

$$\Sigma n = 1 + 2 + ... + n$$
$$= \frac{n(n+1)}{2}$$

(ii) Sum of squares of first n natural numbers

$$\Sigma n^2 = 1^2 + 2^2 + ... + n^2$$
$$= \frac{n(n+1)(2n+1)}{6}$$

(iii) Sum of cubes of first n natural numbers

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + ... + n^3$$
$$= \left(\frac{n(n+1)}{2}\right)^2 = (\Sigma n)^2$$

(iv) Sum of n terms of a sequence $T_n = an^3 + bn^2 + cn + d$

$$\boxed{S_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn}$$

ARITHMETICO-GEOMETRIC SERIES (A.G. S.)

n^{th} term of A.G.. S. = (n^{th} term of an A.P.) × (n^{th} term of a G.P.)

If a, $(a + d)$, $(a + 2d)$ + ... be an A.P. &

b, br, br^2 + ... be a G.P. then

$ab + (a + d)\,br + (a + 2d)br^2$ + ... is the corresponding A.G.S.

$$\boxed{T_n \text{ of A.G.S.} = (T_n \text{ of A.P.}) \times (T_n \text{ of G.P.})}$$

Sum of finite A-G series

$$\Rightarrow \quad S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{(1-r)}$$

For infinite A.G. series

$$\therefore \quad S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}(|r|<1).$$

Method of Difference

When the difference (or difference of differences) of the successive terms of series are in A.P. or G.P, the n^{th} term can be obtained as below. Hence S_n can be found.

Speedy method to Find T_n

If the difference between successive terms of a series are in A.P. then its n^{th} term is of the form $T_n = an^2 + bn + c$ and a, b, c can be found by comparison and hence S_n can be found.