5 CHAPTER Differential Equations

AIEEE Syllabus

Ordinary differential equations, their order and degree. Formation of differential equations. Solution of differential equations by the method of separation of variables. Solution of homogeneous and linear differential equations and those of the type $\frac{d^2y}{dx^2} = f(x)$

An equation involving dependent and independent variables and derivatives of dependent variables is called a ***Differential Equation.*** *It is expressed as :*

$$f\left(a, x, y, z, ..., \frac{dy}{dx}, \frac{d^2y}{dx^2}, ...\right) = 0$$

$$\frac{dy}{dx} = \frac{f(x)}{\phi(y)}, \frac{d^2y}{dx^2} + y\frac{dy}{dx} + y^2 = 0$$

All examples of differential equations

Types of Differential Equations

(1) Ordinary differential equation : A differential equation involving only one independent variable and derivatives w.r.t. this single independent variable is called an ordinary differential equation.

For example : $\frac{dy}{dx} = x + y, \frac{d^4x}{dt^4} + \frac{d^2x}{dt^2}\left(\frac{dx}{dt}\right)^2 = e^t$

(2) Partial Differential Equation : A differential equation involving partial derivatives w.r.t. more than one independent variables is called a partial differential equation.

For example : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(3) Linear and non-linear differential equations : A differential equation is called linear if (i) every dependent variable and every derivative involved occurs in the first degree and (ii) no product of dependent and/or derivatives occur. A differential equation which is not linear is called non-linear differential equations. The general form of a n^{th} order linear differential equation is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

Order and Degree of a Differential Equation

The order of the highest order derivative involved in a differential equation is called the order of the differential equation. The degree of a differential equation is the degree of the highest order derivative which occurs in it, after the differential equation has been made free of radicals and fractions as far as derivatives are concerned.

Note : *We are interested only in linear ordinary differential equations of first order and first degree except some special cases.*

FORMATION OF A DIFFERENTIAL EQUATION

A DE can be formed from a given relation involving dependent and independent variables by the process of differentiation and elimination of parameter.

SOLUTION OF A DIFFERENTIAL EQUATION

Any relation between dependent and independent variables which satisfies the given DE is called solution or integral of the DE. A solution of a DE does not involve the derivatives of dependent variables w.r.t. independent variables.

(1) A solution of DE, consisting of independent arbitrary constants (in number equal to the order of the DE) is called **general solution** or complete primitive

(2) The solution of DE for particular values of one or more of the arbitrary constants is called a **particular solution** of DE.

I. Solution by Inspection :

(i) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{(xy)^2}$

(ii) $d\left(\frac{e^y}{x}\right)=\frac{xe^y dy-e^y dx}{x^2}$

(iii) $d\left(\frac{e^x}{y}\right)=\frac{ye^x dx-e^x dy}{y^2}$

(iv) $d\left\{\frac{1}{2}\log(x^2+y^2)\right\}=\frac{xdx+ydy}{x^2+y^2}$

(v) $d\left(\log\left(\frac{x}{y}\right)\right)=\frac{ydx-xdy}{xy}$

(vi) $d\left(\log\frac{y}{x}\right)=\frac{xdy-ydx}{xy}$

(vii) $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right)=\frac{ydx-xdy}{x^2+y^2}$

(viii) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}$

(ix) $d(xy)=xdy+ydx$

(x) $d(\log(xy))=\frac{xdy+ydx}{xy}$

(xi) $d\left(\frac{x^2}{y^2}\right)=\frac{2y^2xdx-2yx^2dy}{y^4}$

(xii) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydx-2xy^2dy}{x^4}$

(xiii) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(xiv) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

(xv) $d\left(\frac{x^2}{y}\right)=\frac{2xydx-x^2dy}{y^2}$ (xvi) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$

II. (i) **Variable Separable :** In this type of DE it is possible to take all the terms involving x and dx to one side, and involving y and dy to the other side. Thus separating the variables x and y and then integrating. DE can be written as $f(x)\,dx = g(y)\,dy$ or $f_1(x)\,dx + f_2(y)\,dy = 0$

(ii) **Reducible to variable separable :** Sometimes equation are not in the form of variables separable but a suitable substitution can reduce them to variable separable.

III. (i) **Homogeneous DE :** A DE of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous expressions in x and y of the same degree is called a homogeneous equation. It can also be put in the form $\frac{dy}{dx}=F(y/x)$

Working Method :

1st step : Put $y = vx$ so that $\frac{dy}{dx}=v+x\frac{dv}{dx}$

2nd step : Substitute in the given equation and we obtain an equation of the form of variable separable in v and x.

3rd step: After integrating replace v by $\frac{y}{x}$.

(ii) **Reducible to homogeneous form :** Differential equation of type :

$\frac{dy}{dx}=\frac{ax+by+c}{a_1x+b_1y+c_1}$ can be reduced to homogeneous form in the following way :

If $\frac{a}{a_1}\neq\frac{b}{b_1}$ then put $x = X + h$ and $y = Y + k$ where h and k are constants to be determined.

Now equation reduces to

$$\frac{dY}{dX}=\frac{aX+bY+ah+bk+c}{a_1X+b_1Y+a_1h+b_1k+c_1}$$

Now equating $ah + bk + c = a_1h + b_1k + c_1 = 0$, and we have

$$\frac{dY}{dX}=\frac{aX+bY}{a_1X+b_1Y}\quad\text{(homogeneous)}$$

Note : *If* $\frac{a}{a_1}=\frac{b}{b_1}$ *then put* $ax + by = v$ *(equation reduces to variable separable)*

IV. **Linear DE :** A differential equation of the form $\frac{dy}{dx}+Py=Q$ where P and Q are either constant or function of x is called LDE.

Working Method:

1. Write equation in the above form

2. Find integrating factor $IF = e^{\int P dx}$

3. Solution : $ye^{\int P dx} = \int Q e^{\int P dx} dx + c$

Note : If DE is represented as $\frac{dx}{dy} + P_1 x = Q_1$ where P_1 and Q_1 are constant or functions of y. The solution is given by :

$$xe^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} dy + c$$

V. (a) DE reducible to LDE : If equation is of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$ where P and Q are constants, or functions of x then put $f(y) = v$,

$f'(y)\frac{dy}{dx} = \frac{dv}{dx}$ and equation reduces to $\frac{dv}{dx} + Pv = Q$ which is an *LDE* in v and x.

(b) Bernoulli's equation: Equation of the form $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x or constants except 0 and 1. Divide it by y^n we get $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$, now put $y^{1-n} = v$, it reduces to *LDE* in v and x.