3 CHAPTER Complex Numbers

AIEEE Syllabus : Complex numbers as ordered pairs of reals Representation of complex numbers in the form a+ib and their representation in a plane, Argand diagram, algebra of complex numbers, modulus and argument (or amplitude) of a complex number, square root of a complex number, triangle inequality.

IMAGINARY NUMBERS

Square root of a negative number is called imaginary number. While solving the equations $x^2 + 1 = 0$, a quantity $\sqrt{-1}$ is obtaind and is denoted by *i* (iota) which is imaginary.

Further $\sqrt{-2}$ is an imaginary number and can be written as

$$\sqrt{-2} = \sqrt{2} \times \sqrt{-1} \quad = \sqrt{2}\, i$$

If a < 0, then $\sqrt{a} = \sqrt{|a|}\, i$

Integral Powers of *i*

We have $i = \sqrt{-1}$ so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

For any $n \in N$, we have

$$i^{4n+1} = i\ ,\ i^{4n+2} = -1$$

$$i^{4n+3} = -i,\ i^{4n} = 1$$

Thus any integral power of *i* can be expressed as ±1 or ±*i*.

In other words $i^n = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even integer} \\ (-i)^{\frac{n-1}{2}}\, i & \text{if } n \text{ is odd integer} \end{cases}$.

Also $i^{-n} = \dfrac{1}{i^n}$

Illustration 1:

Evaluate:
(i) i^{786}

(ii) $\left(-\sqrt{-1}\right)^{23}$

(iii) $\dfrac{i^2+i^3+i^4+i^5}{i+i^2+i^3}$

Solution :

(i) $i^{786} = i^{4\times196+2} = i^2 = -1$

(ii) $\left(-\sqrt{-1}\right)^{23} = \left(-1\times i\right)^{23} = \left(-i\right)^{23} = -\left(i\right)^{23} = -\left(i\right)^{4\times5+3} = -i^3 = -(-i) = i$

(iii) $\dfrac{i^2+i^3+i^4+i^5}{i+i^2+i^3} = \dfrac{i^2(1+i+i^2+i^3)}{i+i^2+i^3} = \dfrac{(-1)(1+i-1-i)}{i-1-i} = \dfrac{0}{-1} = 0$

Note : *$\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ is true iff atleast one of a and b are non-negative. If a < 0 and b < 0, then*

$\sqrt{a}\times\sqrt{b} = \sqrt{-|a|}\times\sqrt{-|b|}$

$= i\sqrt{|a|}\times i\sqrt{|b|}$

$= -\sqrt{ab}$

COMPLEX NUMBERS

The formal addition, '$a + ib$', where $a, b \in R$ and the collection of all such expressions is called the set of complex numbers. For the complex number, $z = a + ib$, 'a' is called as real part of z and is denoted by $Re(z)$ while 'b' is called as imaginary part of z and is denoted by $\text{Im}(z)$.

Set of complex numbers is denoted by C, which includes the set of real numbers R. *i.e.*, $R \subset C$

A complex number z is said to be purely real if $I_m(z) = 0$ and is said to be purely imaginary if $Re(z) = 0$. The complex number $0 + 0i = 0$ is both purely real and purely imaginary.

All purely imaginary numbers except zero are imaginary numbers but an imaginary number may or may not be purely imaginary.

For *e.g.,* $4 + 3i$ is imaginary but not purely imaginary.

Equality of Complex numbers

Two complex numbers $a + ib$ and $c + id$ are said to be equal, if and only if, $a = c$ and $b = d$. *i.e.*, the corresponding real and imaginary parts are equal.

If $a + ib = C_1$ and $c + id = C_2$

then either $C_1 = C_2$

or $C_1 \neq C_2$

For imaginary numbers, the property of order is not defined because i is neither positive, zero nor negative.

So $C_1 > C_2$ or $C_2 > C_1$ is meaningless till b and d are both equal to zero.

i.e., $C_1 > C_2$ or $C_1 < C_2$ are meaningless if b and d are not equal to zero.

ALGEBRA OF COMPLEX NUMBERS

(i) Addition : $(a + ib) + (c + id) = (a + c) + i(b + d)$

(ii) Subtraction : $(a + ib) - (c + id) = (a - c) + i(b - d)$

(iii) Multiplication : $(a + ib)\cdot(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)$

(iv) Division : $\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$ (When atleast one of c and d is non-zero)

Conjugate Complex Number

For $z = a + ib$, its conjugate is defined as $\bar{z} = a - ib$. Here the complex conjugate is obtained just by changing sign of i.

Properties of conjugate :

(i) $\overline{(\bar{z})} = z$

(ii) $z = \bar{z}$ = iff z is purely real

(iii) $z + \bar{z} = 2Re(z) \Rightarrow Re(z) = Re(\bar{z}) = \frac{z+\bar{z}}{2}$

(iv) $z - \bar{z} = 2i\,Im(z) \Rightarrow I_m(z) = \frac{z-\bar{z}}{2i}$

(v) $z = -\bar{z}$ iff z is purely Imaginary

(vi) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(vii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(ix) $\overline{\left(z^n\right)} = (\bar{z})^n$

(x) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2Re(\bar{z}_1 z_2) = 2Re(z_1 \bar{z}_2)$

(xi) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

Modulus of a Complex Number

The modulus of a complex number $z = x + iy$ is defined as $|z| = \sqrt{x^2+y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$. In other way distance of a complex number z from origin while represented on argand plane is called as modulus of a complex number denoted by mod(z) or $|z|$, or r.

Here $OP = r = \sqrt{x^2+y^2}$

$|z|$ is also called absolute value of z.

Note : *$|z_1 - z_2|$ is the distance between z_1 and z_2.*

Properties of Modulus :

(i) $|z| \geq 0$; $|z| = 0$ iff real and imaginary parts are zero.

(ii) $|z_1 z_2| = |z_1||z_2|$. In general $|z_1 z_2 ......z_n| = |z_1||z_2|....|z_n|$

(iii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, (z_2 \neq 0)$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z\bar{z} = |z|^2$

(vi) $-|z| \le Re(z) \le |z|,\ -|z| \le Im(z) \le |z|$

(vii) $|z^n| = |z|^n$

(viii) $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + 2\text{Re}\,(z_1\bar{z}_2)$

(ix) $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 - 2Re\left(z_1\bar{z}_2\right)$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

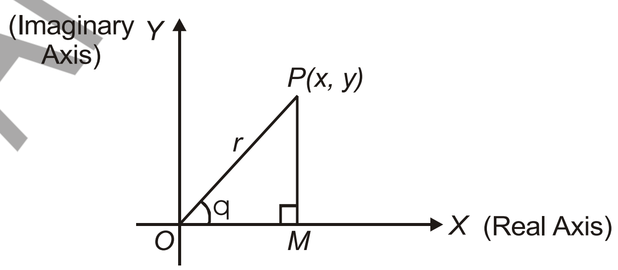
(xi) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in R$

Representation of a Complex Number

Geometrical Representation

The complex number, $z = x + iy$ can be associated with the ordered pair $P(x, y)$.

We consider two perpendicular lines OX and OY (analogous to the Cartesian system) called as the real axis and imaginary axis respectively and where O denotes the origin of reference. The resulting plane is called as Argand plane or Gaussian plane or complex plane and z is represented by the point P corresponding to the ordered pair (x, y). The point P is called as affix of z.



Argument or Amplitude of z

The argument of z, denoted by arg z or amp z is the angle which OP makes with the positive direction of real axis, the angle being measured in anticlockwise sense.

If $z = x + iy$ then the angle θ given by $\tan\theta = \frac{y}{x}$ is said to be the argument of z.

or $\arg(z) = \text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$

Note : *Arg (z) is not unique; if θ_l is one value, then $\theta_1 + 2k\pi \ : \ k \in I$ is the set of all possible values of arg z. Any two arguments of a complex number differ by $2k\pi$.*

Principal argument of a complex number z is that value of argument (z) which lies in the interval $(-\pi, \pi]$.

For $z = x + iy$, $\theta = \tan^{-1}\left|\frac{y}{x}\right|$, then principal argument depends on the quadrant in which point (x, y) lies.

(i) $x > 0, y = 0$ $\Rightarrow$ Principal argument = 0 (Positive Real Axis)

(ii) $x > 0, y > 0$ $\Rightarrow$ Prinicpal argument = θ (I[st] quadrant)

(iii) $x = 0, y > 0$ $\Rightarrow$ Prinicpal argument = $\frac{\pi}{2}$ (Positive imaginary axis)

(iv) $x < 0, y > 0$ $\Rightarrow$ Principal argument = $\pi - \theta$ (II[nd] quadrant)

(v) $x < 0, y = 0$ $\Rightarrow$ Prinicpal argument = π (Negative Real Axis)

(vi) $x < 0, y < 0$ $\Rightarrow$ Principal argument = $-\pi + \theta$ (III[rd] quadrant)

(vii) $x = 0, y < 0$ $\Rightarrow$ Principal argument = $-\frac{\pi}{2}$ (Negative Imaginary Axis)

(viii) $x > 0, y < 0$ $\Rightarrow$ Principal argument = $-\theta$ (iv[th] quadrant)

Properties of Argument

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \ : \ k = 0, \pm 1$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi \ : \ k = 0, \pm 1$

(iii) $\arg(z^2) = 2\arg(z) + 2k\pi \ : \ k = 0, \pm 1$

(iv) If arg (z) = 0 or π then z is real

(v) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi : \ k = 0, \pm 1$

(vi) $\arg(z^n) = n\arg(z) + 2k\pi \ : \ k = 0, \pm 1$

(vii) If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta; k \in I$

(viii) $\arg(\bar{z}) = -\arg(z)$

(ix) $\arg(z - \bar{z}) = \pm\frac{\pi}{2}$

(x) $\arg(z) - \arg(-z) = \pm\pi$

Polar form (Trigonometric form) of a Complex Number

Let $OP = r$, then $x = r\cos\theta$, and $y = r\sin\theta$

$\Rightarrow$ $z = x + iy = r\cos\theta + ir\sin\theta$, $= r(\cos\theta + i\sin\theta)$. This is known as Trigonometric (or Polar) form of a complex Number. Here we should take the principal value of θ.

For general values of the argument

$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ (where n is an integer)

Note : *Sometimes $\cos\theta + i\sin\theta$, in short is written as $cis(\theta)$.*

Exponential form of a Complex Number (Euler's Form)

According to Euler's Theorem, $e^{i\theta} = \cos\theta + i\sin\theta$ and therefore $z = r(\cos\theta + i\sin\theta)$ can be written as $z = re^{i\theta}$ which is called as exponential form of a complex number.

Replacing θ by $-\theta$ in $e^{i\theta}$, we obtain

$e^{-i\theta} = \cos\theta - i\sin\theta$

Hence $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{x}{\sqrt{x^2 + y^2}}$

and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{y}{\sqrt{x^2 + y^2}}$

Illustration 2 :

Write the following complex numbers in polar and exponential form

(1) $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ (2) $1 - i$

Solution :

(1) Here $r\cos\theta = -\dfrac{1}{2}$, $r\sin\theta = -\dfrac{\sqrt{3}}{2}$

Squaring and adding $r^2\cos^2\theta + r^2\sin^2\theta = \dfrac{1}{4} + \dfrac{3}{4} = 1$

$\therefore$ $r = 1$ (–ve value is rejected)

Dividing we get $\tan\theta = \sqrt{3} = \tan\dfrac{\pi}{3}$

Since $\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$ lies in third quadrant

Principal argument $= \dfrac{\pi}{3} - \pi = -\dfrac{2\pi}{3}$

$\therefore$ Polar form of $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ is $1\left(\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right)$ and Euler's form is $1.e^{-\frac{2\pi}{3}i}$

(2) Here $r\cos\theta = 1$ and $r\sin\theta = -1$

$\therefore\ r = \sqrt{2}$, $\tan\theta = -1 = \tan\left(-\frac{\pi}{4}\right)$

Since (1, –1) lies in IVth quadrant, principal value of θ is $-\frac{\pi}{4}$

$\therefore$ Polar form of $1 - i$ is $\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

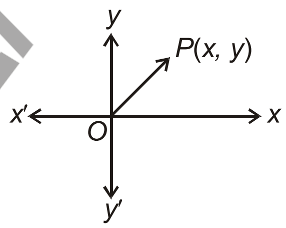
$= \sqrt{2}.e^{-\frac{i\pi}{4}}$

Remember : $1 = e^{i0}$, $i = e^{\frac{i\pi}{2}}$, $-i = e^{-\frac{i\pi}{2}}$, $-1 = e^{i\pi}$

$$\log i = \log e^{\frac{i\pi}{2}} = i\frac{\pi}{2};\ \log(\log i) = \log\left(i\frac{\pi}{2}\right) = \log i + \log\frac{\pi}{2} = i\frac{\pi}{2} + \log\frac{\pi}{2}$$

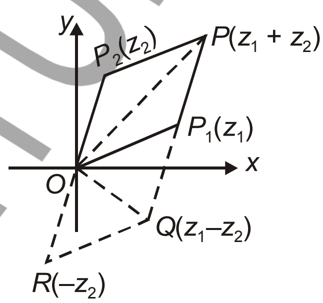
Vector representation of a Complex Number

A complex number $z = x + iy$ can be represented by the position vector OP of point $P(x, y)$ in a two dimensional plane becuase a complex number depends on two things viz (i) its modulus and (ii) its argument which are also the requirements of a vector on a plane.



Geometrical meaning of Algebraic Operations

(i) Let $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ be represented by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively.



Then, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ is represented by the point P (where $OP_1 P P_2$ is a parallelogram).

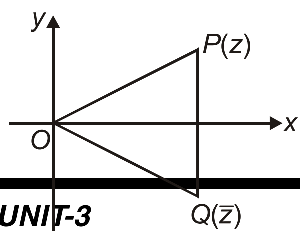
Similarly $z_1 - z_2 = z_1 + (-z_2)$ is represented by the point Q (as shown).

In vector notation $z_1 + z_2 = \overrightarrow{OP_1} + \overrightarrow{OP_2} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = \overrightarrow{OP}$

and $z_1 - z_2 = \overrightarrow{OP_1} - \overrightarrow{OP_2}$

$= \overrightarrow{OP_1} + \overrightarrow{OR} = \overrightarrow{OQ}$

(ii) If $z = x + iy$, then, the conjugate of z, denoted by $\bar{z}$ is the number $\bar{z} = x - i\,y$,



which is the mirror image of z along real axis.

Note :

1. *If z lies in 2nd quadrant, $\bar{z}$ lies in 3rd quadrant, and if z lies in 1st quadrant, $\bar{z}$ lies in 4th quadrant.*
2. *If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be represented by point Q_1 (x_1, y_1) and Q_2 (x_2, y_2) respectively then $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ is represented by the point P (as shown)*
3. *In polar form, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then the multiplication $z_1z_2 = r_1r_2e^{i(\theta_1\theta_2)}$ increases or decreases the modulus of z_2 by r_1 times and rotates z_2 anticlockwise by an angle equal to argument of z_1 about an axis passing through its tail and perpendicular to the argand plane in which the two complex vectors are lying.*
4. *Multiplication of a complex number $z(= re^{i\theta})$ by i rotates z anticlockwise about origin by $\frac{\pi}{2}$ and mulitplication of z by –i rotates z clockwise about origin by $\frac{\pi}{2}$.*

TRIANGLE INEQUALITY

In any triangle, sum of any two sides is greater than the third side and difference of any two sides is less than the third side, we have

(i) $|z_1| + |z_2| \geq |z_1 + z_2|$; here equality holds when $\arg\left(\frac{z_1}{z_2}\right) = 0$ *i.e.*, position vectors representing two complex numbers z_1 and z_2 are parallel.

(ii) $||z_1| - |z_2|| \leq |z_1 - z_2|$; here equality holds when $\arg\left(\frac{z_1}{z_2}\right) = 0$ *i.e.*, position vectors representing two complex numbers z_1 and z_2 are parallel.

Parallelogram law :

In the parallelogram OP_1PP_2, the sum of the squares of its sides is half of the sum of the squares of its diagonals

$$\Rightarrow \; |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

Square Root of a Complex Number:

Let $z = x + iy$ be a complex number such that $\sqrt{x + iy} = a + ib$ where $x, y \in R$

$\sqrt{x+iy}$ **=** $a + ib \;\Rightarrow\; (a + ib)^2 = x + iy$

$\Rightarrow\; a^2 - b^2 + 2abi = x + iy$

On equating real and imaginary parts,

$x = a^2 - b^2$ ......(i)

and $y = 2ab$ ......(ii)

Solving (i) and (ii), we get $a = \pm\sqrt{\frac{1}{2}(x + \sqrt{x^2 + y^2})}$

and $b = \pm\sqrt{\frac{1}{2}(\sqrt{x^2 + y^2} - x)}$

Hence $$\sqrt{x + iy} = \begin{cases} \pm\left[\sqrt{\dfrac{|z| + x}{2}} + i\sqrt{\dfrac{|z| - x}{2}}\right] & \text{for } y > 0 \\ \pm\left[\sqrt{\dfrac{|z| + x}{2}} - i\sqrt{\dfrac{|z| - x}{2}}\right] & \text{for } y < 0 \end{cases}$$

Note : *This result need not to be memorised.*

Remember : $\sqrt{i} = \pm\left(\frac{1 + i}{\sqrt{2}}\right), \sqrt{-i} = \pm\left(\frac{1 - i}{\sqrt{2}}\right)$

Application of Complex Numbers to Geometrical Application

1. Distance Formula :

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex number represented by points P and Q on the argand plane.

Then the distance $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Also $z_2 - z_1 = (x_2 + iy_2) - (x_1 + iy_1) = (x_2 - x_2) + i(y_2 - y_1)$

$\therefore \quad |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = PQ$

Hence distance between two points z_1 and z_2 is given by $|z_2 - z_1|$

Illustration 3 :

If z_1 and z_2 are two fixed points in the argand plane then find the locus of a point z in each of the following

(i) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(ii) $|z - z_1| = |z - z_2|$

(iii) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

Solution :

(i) Let P and Q be two points represented by z_1 and z_2 in the argand plane and let R be a point having affix z.

Then $PR = |z - z_1|$, $QR = |z - z_2|$ and $PQ = |z_1 - z_2|$

$|z - z_1| + |z - z_2| = |z_1 - z_2|$

$\Rightarrow PR + QR = PQ$

$\Rightarrow$ $R(z)$ lies on the line segment joining $P(z_1)$ and $Q(z_2)$

(ii) $|z - z_1| = |z - z_2|$

$\Rightarrow$ $PR = QR$

$\Rightarrow$ $R(z)$ is equidistant from $P(z_1)$ and $Q(z_2)$.

$\Rightarrow$ R lies on the perpendicular bisector of line segment PQ.

(iii) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

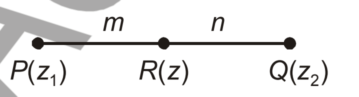
$\Rightarrow$ $PR - QR = PQ$

$\Rightarrow$ $R(z)$ lies on the line joining $P(z_1)$ and $Q(z_1)$ but does not lie between them.

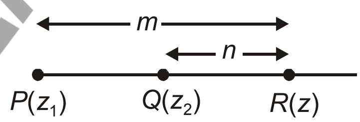
2. Section formula :

If the point R(z) divides the line segment PQ (where P and Q represent the complex numbers z_1 and z_2 respectively)

(i) Internally in the ratio $m : n$ then, $z = \dfrac{mz_2 + nz_1}{m+n}$



(ii) Externally in the ratio $m : n$ then, $z = \dfrac{mz_2 - nz_1}{m-n}$



(iii) Mid point of line segment PQ = $\dfrac{z_1 + z_2}{2}$

3. Area of Triangle

If the vertices of a triangle ABC represent the complex numbers z_1, z_2 and z_3 respectively then area of the triangle is the modulus of $\dfrac{i}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$.

Illustration 4 :

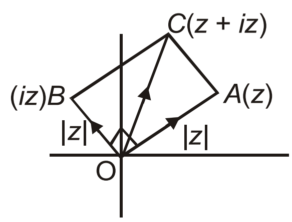
Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $\dfrac{1}{2}|z|^2$.

Solution :

Let $z = x + iy$

then co-ordinates in ordered pair of z and iz are (x, y) and $(-y, x)$.

Here origin 0, z, $z + iz$ and iz form a square of side $|z|$.



Hence area of the required triangle is $\dfrac{1}{2}|z|^2$. (Half the area of the square)

4. Slope of line segment joining two points

If P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the complex slope of PQ is defined to be $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

5. Condition for collinearity: Three points $A(z_1)$ $B(z_2)$ and $C(z_3)$ will be collinear

(i) If there exists a relation $az_1 + bz_2 + cz_3 = 0$, such that $a + b + c = 0$ (where a, b, c are non-zero real numbers) (where a, b, c are not all zero real numbers)

(ii) If the area of ΔABC formed by z_1, z_2 and z_3 is zero

$$i.e., \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

(iii) If slope of AB = slope of BC = slope of AC

$$i.e., \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

6. Equation of a Straight Line

Let $P(z_1)$ and $Q(z_2)$ be two given points in the Argand plane.

If $R(z)$ be any point on line joining P and Q, then we have

$\arg(z - z_2) = \arg(z_2 - z_1)$

$$i.e., \arg\left(\frac{z - z_2}{z_2 - z_1}\right) = 0$$

$$\Rightarrow \frac{z - z_2}{z_2 - z_1} - \overline{\left(\frac{z - z_2}{z_2 - z_1}\right)} = 0$$ (If $\arg(z) = 0$ then z is real $i.e.$, $z - \bar{z} = 0$)

$$\Rightarrow \frac{z - z_2}{z_2 - z_1} - \frac{\bar{z} - \bar{z}_2}{\bar{z}_2 - \bar{z}_1} = 0$$

or $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - z_2\bar{z}_1) = 0$ ....(i)

which can also be written as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$ ....(ii)

If in equation (i), $\bar{z}_1 - \bar{z}_2 = \bar{a}$ then $z_1 - z_2 = a$

$\Rightarrow$ Equation of straight line is of the form

$z\bar{a} - \bar{z}a + b = 0$ ....(iii)

where $b = z_1\bar{z}_2 - z_2\bar{z}_1$, which is purely imaginary.

However, if equation (iii) is multiplied by *i*, it reduces to

$$z(i\bar{a}) - \bar{z}(ia) + (ib) = 0$$

or $z(\bar{A}) + \bar{z}A + B = 0$ .....(iv)

($\because -ia,\ i\bar{a}$ are conjugates)

where $B = ib$ is purely real

Hence any equation of form (i), (ii), (iii) or (iv) will represent a straight line.

Examples :

(i) $3z + 3\bar{z} + 2 = 0$ is a straight line since it is of form (iv)

(ii) $3z - 3\bar{z} - 6i = 0$ is also a stright line of form (iii).

(iii) $3z + 3\bar{z} + 6i = 0$ is not a straight line.

For the straight line $\bar{a}z + a\bar{z} + b = 0,\ b \in R$

Real slope of the line = $\dfrac{a + \bar{a}}{i(a - \bar{a})} = -\dfrac{Re(a)}{Im(a)}$

and complex slope = $-\dfrac{\bar{a}}{a}$

7. Distance of a point from a given line

Distance of a point $P(z_1)$ from line $z\bar{a} + \bar{z}a + b = 0$ is given by

$$\frac{|z_1\bar{a} + \bar{z}_1 a + b|}{\sqrt{(Re(a))^2 + (Im(a))^2}} = \frac{|z_1\bar{a} + \bar{z}_1 a + b|}{2|a|}$$

8. Condition for perpendicular or parallel lines

Let complex slope of two lines be α_1 and α_2

(i) For perpendicular lines, $\alpha_1 + \alpha_2 = 0$

(ii) For parallel lines, $\alpha_1 = \alpha_2$

$\therefore$ Lines $\bar{a}z + a\bar{z} + k_1 = 0$ and $\bar{b}z + b\bar{z} + k_2 = 0$

$(k_1, k_2 \in R)$ are perpendicular if $-\dfrac{\bar{a}}{a} + \left(-\dfrac{\bar{b}}{b}\right) = 0$

or $\bar{a}b + a\bar{b} = 0$

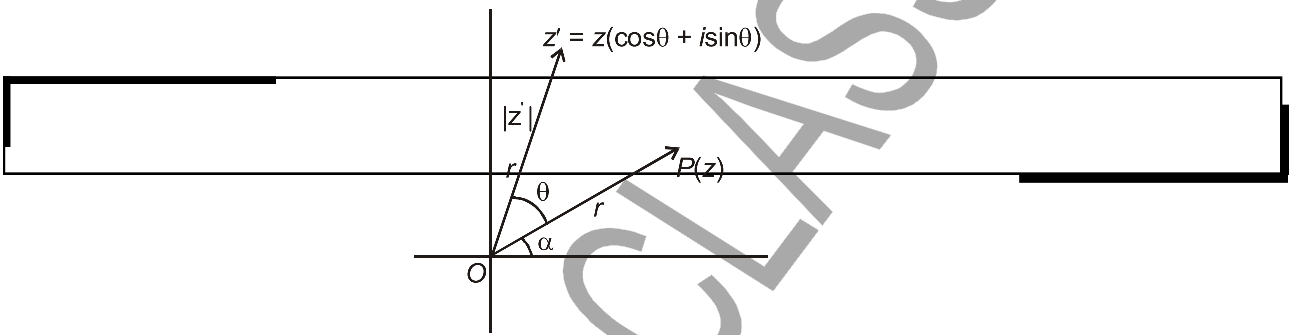
and parallel if $-\frac{\bar{a}}{a} = -\frac{\bar{b}}{b}$ or $\bar{a}b = a\bar{b}$

Note : *Line parallel to* $z\bar{a} + \bar{z}a + b = 0$ *is given by* $z\bar{a} + \bar{z}a + \lambda = 0$ $(\lambda \in R)$

Line perpendicualr to $z\bar{a} + \bar{z}a + b = 0$ *is given by* $z\bar{a} - \bar{z}a + i\lambda = 0$ $(\lambda \in R)$

9. Concept of rotation

Multiplication of a complex number *z* represented by point *P*, with $e^{i\theta} = (\cos\theta + i\sin\theta)$ rotates the line *OP* by an angle θ, anticlockwise about *O*.

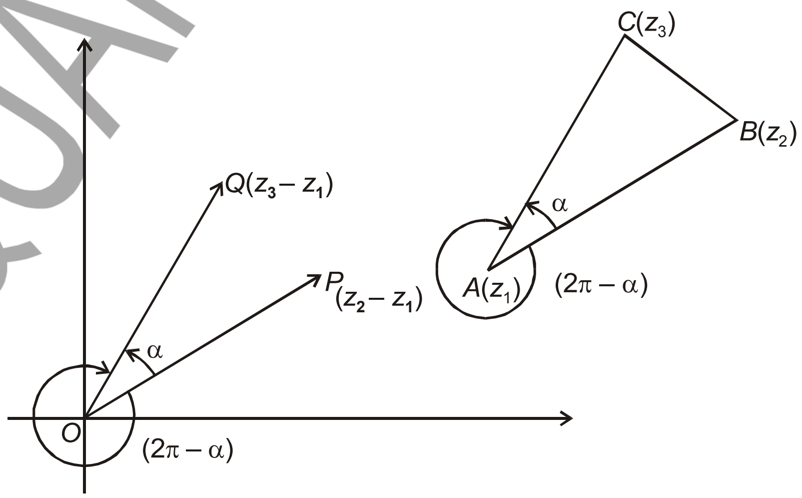


Let z_1, z_2, z_3 be the affixes of the vertices of a ΔABC described in the counter-clockwise sense. Draw *OP* and *OQ* parallel and equal to *AB* and *AC* respectively. Then the point *P* is $z_2 - z_1$ and *Q* is $z_3 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}(\cos\alpha + i\sin\alpha)$$

$$= \frac{CA}{BA}e^{i\alpha} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\alpha}$$

or $\text{amp}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$



Note : We can also write $= \dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|} e^{-i(2\pi - \alpha)}$

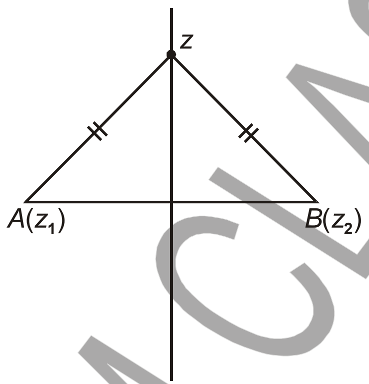
In this case, we are rotating $\overrightarrow{OP}$ in clockwise direction by an angle $(2\pi - \alpha)$. Since the rotation is in clockwise direction, we are taking negative sign with angle $(2\pi - \alpha)$.

10. Equation of the Perpendicular Bisector

The equation of perpendicular bisector of the line segment joining points $A(z_1)$ and $B(z_2)$ is

$$|z - z_1| = |z - z_2|$$

or $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$



11. Properties of a triangle

Let $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of a triangle, then :

(i) Centroid $\dfrac{z_1 + z_2 + z_3}{3}$

(ii) If ABC is an equilateral triangle, then the circumcentre (z_0) satisfies the relation

$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

(iii) Incentre = $\dfrac{z_1|z_2 - z_3| + z_2|z_3 - z_1| + z_3|z_1 - z_2|}{|z_2 - z_3| + |z_3 - z_1| + |z_1 - z_2|}$

(iv) If *ABC* is an equilateral triangle than

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

or $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$

(v) If *ABC* is an isosceles right-angled triangle right angled at *B* then

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

and $z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$

12. Circle:

(a) Circle with center represented by the complex number z_0 and radius r has the equation $|z-z_0| = r$.

(b) The general equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where b is a real number. The center of this circle is '$-a$' and its radius is $\sqrt{a\bar{a} - b}$.

(c) The equation of the circle described on the line segment joining z_1 and z_2 as diameter is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0.$$

(d) The point with affix z_1 lies inside (resp. outside) the circle $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ if

$z_1\bar{z}_1 + a\bar{z}_1 + \bar{a}z_1 + b < 0$ (resp. > 0).

(e) If z is a variable point such that $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (where $\alpha \neq n\pi$, α is a constant) then z describes an arc of a circle.

(i) $\left|\frac{z - z_1}{z - z_2}\right| = k$ is a circle, if $k \neq 1$ and is a line, if $k = 1$.

(ii) The equation $|z - z_1|^2 + |z - z_2|^2 = k$ represents a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$.

(iii) If $\arg\left[\frac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\right] = \pm\pi, 0$, then the points z_1, z_2, z_3 are concyclic.

(iv) $|z - z_0| < r$ represents interior of the circle $|z - z_0| = r$ and $|z - z_0| > r$ represents exterior of the circle $|z - z_0| = r$

Useful Result

If $\left|z + \frac{1}{z}\right| = a$, the greatest and least valves of $|z|$ are respectively $\frac{a + \sqrt{a^2 + 4}}{2}$ and $\frac{-a + \sqrt{a^2 + 4}}{2}$

SOLVED EXAMPLES

Example 1 :

Find the locus of complex number z if $\left|\frac{1 - iz}{z - i}\right| = 1$.

Solution :

Given $\left|\dfrac{1-iz}{z-i}\right| = 1$

$\Rightarrow \quad \left|\dfrac{i^4 + i^3 z}{z-i}\right| = 1 \Rightarrow |\, i^3 \,| \left|\dfrac{z+i}{z-i}\right| = 1$

$\Rightarrow \quad |z + i| = |z - i|$

Which represents the equation of perpendicular bisector of i and $-i$ and that will be the x-axis.

Example 2 :

If f(z) is divided by (z – i) and (z + i), the remainders are respectively i and 1 + i. Determine the remainder when f(z) is divided by $z^2 + 1$.

Solution :

$f(z) = (z-i)Q_1(z) + i$ and $f(z) = (z+i)Q_2(z) + 1 + i$

where $Q_1(z)$ and $Q_2(z)$ are the quotients where $f(z)$ is divided by $(z - i)$ and $(z + i)$ respectively.

$\therefore \quad f(i) = i$ and $f(-i) = 1 + i$

We have to determine the remainder when $f(z)$ is divided by $z^2 + 1$. Since the divisor is of degree 2, the remainder must be of degree less than 2 (i.e. either 1 or 0).

Then $\quad f(z) \quad = (z^2 + 1)Q(z) + pz + q \;:\; p, q \in C.$

$\Rightarrow \quad f(z) = (z - i)(z + i)Q(z) + pz + q$

$pi + q = f(i) = i \qquad \text{... (1)}$

$-pi + q = f(-i) = 1 + i \qquad \text{... (2)}$

Solving (1) and (2) for p and q, we have :

$$p = \frac{i}{2}, \quad q = \frac{1+2i}{2}$$

Hence, the remainder is $\dfrac{iz}{2} + \dfrac{1+2i}{2}$.

Example 3 :

Find the least and greatest value of |z| which satisfies $\left|z + \dfrac{1}{z}\right| = a$, where $a \in R$.

Solution :

Let $\quad z = r(\cos\theta + i\sin\theta)$ then $\dfrac{1}{z} = \dfrac{1}{r}(\cos\theta - i\sin\theta)$

$$a = \left|z + \frac{1}{z}\right| = \left|z - \left(-\frac{1}{z}\right)\right| \geq \left||z| - \frac{1}{|z|}\right|$$

$$-a \le r - \frac{1}{r} \le a$$

$$r^2 + ar - 1 \ge 0 \quad \text{and} \quad r^2 - ar - 1 \le 0$$

$$\left(r \ge \frac{-a+\sqrt{a^2+4}}{2} \quad \text{or} \quad r \le \frac{-a-\sqrt{a^2+4}}{2}\right) \quad \text{and} \quad \frac{a-\sqrt{a^2+4}}{2} \le r \le \frac{a+\sqrt{a^2+4}}{2}$$

$$\therefore \quad \frac{-a+\sqrt{a^2+4}}{2} \le r \le \frac{a+\sqrt{a^2+4}}{2}$$

$$r_{\max.} = \frac{a+\sqrt{a^2+4}}{2} \; ; \qquad r_{\min.} = \frac{-a+\sqrt{a^2+4}}{2} .$$

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