14 CHAPTER

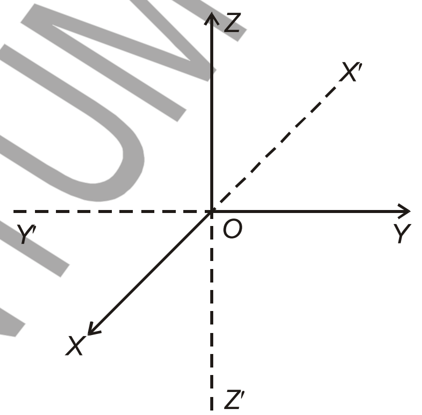
Three - Dimensional Geometry

AIEEE Syllabus : Coordinates of a point in space, distance between two points, Section formula, direction ratios and direction cosines, angle between two intersecting lines. Skew lines, the shortest distance between them and its equation. Equations of a line and a plane in different forms; intersection of a line and a plane, coplanar lines.

In the practical life different objects do not lie in same plane but in space. If we are to locate any object in universe three co-ordinates are required. Hence, three co-ordinate axes namely X, Y & Z intersecting at a point, called origin are chosen in mutually perpendicular directions. In terms of 3 co-ordinates i.e. (X, Y, Z) any point in universe can be specified exactly.

CO-ORDINATE AXES AND ORIGIN

Let XOX', YOY' and ZOZ' are three mutually perpendicular lines intersecting at O. 'O' is called origin of co-ordinate system.



OX → Positive direction of x-axis
OX' → Negative direction of x-axis
OY → Positive direction of y-axis
OY' → Negative direction of y-axis
OZ → Positive direction of z-axis
OZ' → Negative direction of z-axis

DISTANCE FORMULA

Distance PQ in cartesian co-ordinates where P is (x_1, y_1, z_1) and Q is (x_2, y_2, z_2)

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad \text{... (i)}$$

In vector form

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\therefore |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \text{same as ...(i)}$$

Note : Distance of P (x_1, y_1, z_1) from origin O (0, 0, 0) = $\sqrt{x_1^2 + y_1^2 + z_1^2}$

$$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$|\overrightarrow{OP}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

SECTION FORMULA

I. Internal division

If point $R(x, y, z)$ divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m : n$ then

$$x = \frac{mx_2 + nx_1}{m + n}$$

$$y = \frac{my_2 + ny_1}{m + n}$$

$$z = \frac{mz_2 + nz_1}{m + n}$$

$$\therefore R = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

Vectorially ; $\overrightarrow{OR} = \dfrac{m\overrightarrow{OQ} + n\overrightarrow{OP}}{m + n}$

II. External division

If $R(x, y, z)$ divides PQ externally in $m : n$ then

$$R = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

Vectorially ; $\overrightarrow{OR} = \dfrac{m\overrightarrow{OQ} - n\overrightarrow{OP}}{m - n}$

Mid point is given by $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

Centroid of the Triangle

If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are the vertices of a triangle, then centroid is given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

Centroid of a Tetrahedron

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then the centroid G of tetrahedron is given by $\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$.

Direction-Cosines of a Line

If α, β, γ are the angles that a given line makes with the positive directions of the co-ordinate axes, then cos α,
cos β, cos γ are called the Direction-cosines (or d.c.'s) of the given line.

Usually the direction cosines of a line are denoted by l, m, n. Thus [l, m, n] denote d.c.'s of a line.

Cor. Direction-cosines of Co-ordinate Axes : The x-axis makes angles of 0°, 90° and 90° with axes of X, Y and Z respectively. Hence direction-cosines of X-axis are [cos 0°, cos 90°, cos 90°], i.e., [1, 0, 0].

Similarly d.c.'s of Y-axis are [0, 1, 0] and those of Z-axis are [0, 0, 1].

Position of a point by Radius Vector and Direction-Cosines

Let $P(x, y, z)$ be a point in the space and O the origin. Then length $OP = \vec{r}$ is the radius vector of the point P.

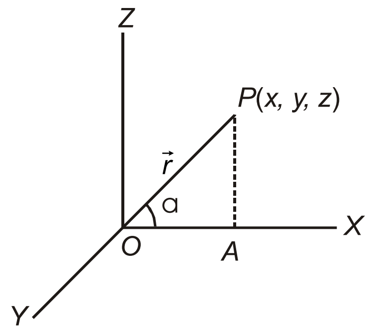
Also let (cos α, cos β, cos γ) be the d.c.'s of OP. Drop PA perpendicular to OX. Then

$x = OA = OP \cos\alpha = r\cos\alpha$,

i.e., $x = lr$ if $\cos\alpha = l$.

Similarly $y = mr$, $z = nr$.

Hence co-ordinates of P are (lr, mr, nr).



Note : In vector notations $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Since $\cos\alpha = \frac{x}{r}, \cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$.

Relation between Direction-Cosines

If l, m, n are the direction-cosines of a line then $l^2 + m^2 + n^2 = 1$.

Direction-Ratios : Quantities proportional to direction-cosines of a line are called direction-ratios of that line. Thus if l, m, n are direction-cosines and a, b, c direction-ratios of the line, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

as a, b, c are proportional to l, m, n respectively.

Thus $l = \frac{\pm a}{\sqrt{a^2+b^2+c^2}}$ etc.

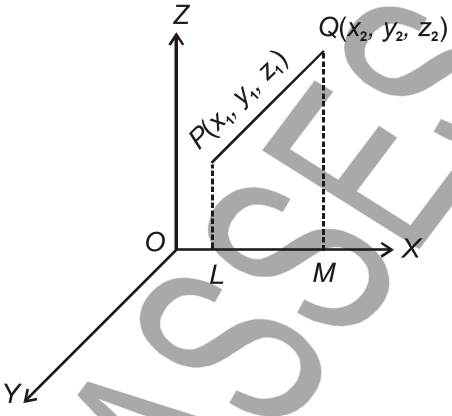
Hence if a, b, c are direction-ratios, then actual direction-cosines are

$$\left(\frac{\pm a}{\sqrt{a^2+b^2+c^2}}, \frac{\pm b}{\sqrt{a^2+b^2+c^2}}, \frac{\pm c}{\sqrt{a^2+b^2+c^2}}\right)$$

Direction-cosines of line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

Let l, m, n be the direction-cosines of PQ; then projection of PQ on X-axis = $PQ \cdot l$

$PQ \cdot l = x_2 - x_1$



$$\therefore \quad l = \frac{x_2 - x_1}{PQ}$$

Similarly $m = \dfrac{y_2 - y_1}{PQ}$

and $n = \dfrac{z_2 - z_1}{PQ}$

$$\therefore \quad \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ$$

Therefore direction-cosines of line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

EQUATION OF A STRAIGHT LINE

1. Vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to the vector $\vec{b}$ is

   $\vec{r} = \vec{a} + t\vec{b}$ **(One Point Form)**

2. **Cartesian Form :**

   Let $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{b} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

   This gives $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} (= t)$.

3. Vector equation of a line through the point $A(\vec{a})$ and $B(\vec{b})$ is

   $$\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) \quad \textbf{(Two Point Form)}$$

4. If $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then parametric form are

   $x = x_1 + t(x_2 - x_1)$, $y = y_1 + t(y_2 - y_1)$, $z = z_1 + t(z_2 - z_1)$

   Cartesian form $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

ANGLE BETWEEN TWO LINES

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

where θ is the angle between the lines whose direction-cosines are $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$.

Perpendicular and Parallel Lines

Two lines whose direction-cosines are $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ are perpendicular if

$l_1l_2 + m_1m_2 + n_1n_2 = 0$ and parallel if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note : Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are **coplanar** iff

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Shortest Distance between Two Skew Lines

1. The shortest distance between two non-parallel lines $\vec{r} = \vec{a_1} + t\vec{b_1}$ and $\vec{r} = \vec{a_2} + s\vec{b_2}$ is

$$\left| \frac{\left(\vec{b_1} \times \vec{b_2}\right) \cdot \left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|} \right|$$

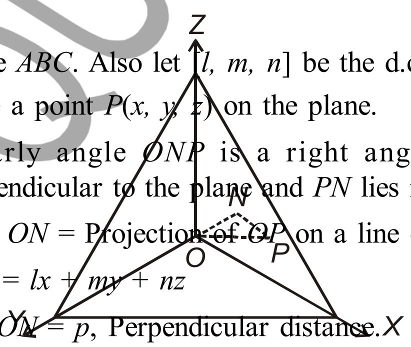
Cor. These lines are coplanar iff $[\vec{a}_2 - \vec{a}_1 \;\; \vec{b}_1 - \vec{b}_2] = 0$

2. The distance (i.e., the shortest distance) between two parallel lines $\vec{r} = \vec{a_1} + t\vec{b}$ and $\vec{r} = \vec{a_2} + s\vec{b}$ is

$$\frac{\left|\vec{b} \times \left(\vec{a_2} - \vec{a_1}\right)\right|}{\left|\vec{b}\right|}$$

THE PLANE

1. Every equation of the first degree in x, y, z represents a plane i.e., $ax + by + cz + d = 0$
2. $\vec{r} \cdot \hat{n} = P$ is the vector equation of a plane.
3. **Intercept Form :** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ represents a plane and that its intercepts on the three axes of co-ordinates are a, b, c respectively.
4. **Normal Form :** Let $ON = p$ be the perpendicular from O on the plane ABC. Also let $[l, m, n]$ be the d.c.'s of normal ON.



Take a point $P(x, y, z)$ on the plane.

Clearly angle ONP is a right angle because ON is perpendicular to the plane and PN lies in the plane.

Also ON = Projection of OP on a line ON

$= lx + my + nz$

But $ON = p$, Perpendicular distance.

Hence $lx + my + nz = p$ is the required equation of the plane.

5. **One–Point Form :** Vector equation of the plane through the point $A(\vec{a})$ and perpendicular to the direction $\vec{n}$ is

   $(\vec{r}-\vec{a}).\vec{n} = 0$

6. Equation of the plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad \textbf{(Three Point form)}$$

Angle between Two Planes

1. If θ is the angle between the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$, then

   $$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

   where $\vec{n}_1$ and $\vec{n}_2$ are vectors along normals to the two planes.

2. If $\vec{n}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

   and $\vec{n}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

   $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

3. Two planes are perpendicular iff

   $a_1a_2 + b_1b_2 + c_1c_2 = 0$ & Parallel if $f\ \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Angle between a Line and a Plane

$$\sin\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}},$$

where (a, b, c) are the direction rations of a line perpendicular to the plane and (l, m, n) are the direction ratios of the line.

This gives the angle between the line and the plane.

1. If line is parallel to the plane, θ = 0 (or the line is perpendicular to the normal AN),

   ∴ $al + bm + cn = 0$

2. The line will be perpendicular to the plane, if it is parallel to the normal to the plane, *i.e,* $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$.

Perpendicular Distance of a Point from a Plane

1. **Plane in the normal form :**

   Let the equation of the plane be

$x \cos \alpha + y \cos \beta + z \cos \gamma = P$ ...(1)

P being the perpendicular distance of the plane from origin

Any plane parallel to (1) is

$x \cos \alpha + y \cos \beta + z \cos \gamma = P'$ ...(2)

It passes through (x_1, y_1, z_1)

$\therefore \quad x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma = P'$ ...(3)

Now perpendicular distance of the point (x_1, y_1, z_1) from (1) = $P' - P$

$= x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - P$

2. **Plane given by general equation.** Let the equation of the plane be

$ax + by + cz + d = 0$

Changing it into normal form the equation becomes $\dfrac{ax + by + cz + d}{\pm\sqrt{(a^2 + b^2 + c^2)}} = 0$ ...(1)

Perpendicular distance of the (x_1, y_1, z_1) from (1)

$$= \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}} \right|$$

PLANE THROUGH THE INTERSECTION OF TWO PLANES

1. Equation of any plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(\lambda\vec{n}_1 + \mu\vec{n}_2) = \lambda d_1 + \mu d_2, \lambda^2 + \mu^2 \neq 0$

2. Equation of any plane through the intersection of the planes

   $P \equiv a_1x + b_1y + c_1z + d_1 = 0$

   and $\quad Q \equiv a_2x + b_2y + c_2z + d_2 = 0$

   Then the equation $P + \lambda Q = 0$

   i.e., $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$