12
CHAPTER

Conic Sections

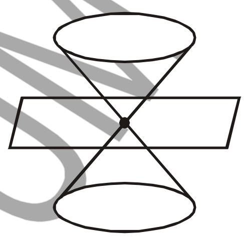
AIEEE Syllabus : Sections of cones, equations of conic sections in standard forms. Condition for y = mx + c to be a tangent and point (s) of tangency

Conic sections, namely a point, pair of straight lines, circle, ellipse, parabola and hyperbola are called so because they can be obtained when a cone (or double cone) is cut by a plane.

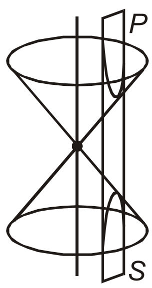
The mathematicians associated with the study of conics were Euclid, Aristarchus and Apollonius. Most of the objects around us and in space have shape of conic-sections. Hence study of these becomes a very important tool for present knowledge and further exploration.

Conic-sections as Sections of Right Circular Cone(s)

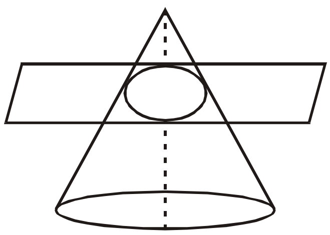
1. When a double right circular cone is cut by a plane parallel to base at the common vertex, the cutting profile is a point.



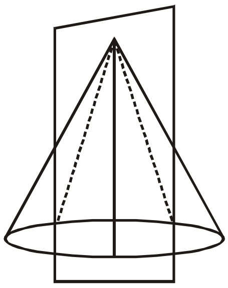
2. When a double right circular cone is cut by plane, parallel to its common axis, the cut profile is hyperbola



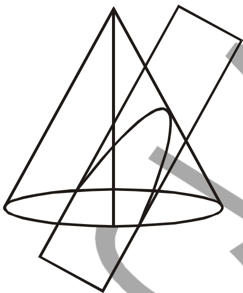
3. When a right circular cone is cut by a plane parallel to its base the cutting profile is a circle.



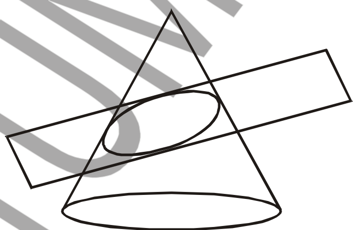
4. When a right circular cone is cut by any plane through its vertex, the cutting profile is a pair of straight lines through its vertex



5. When a right circular cone is cut by a plane parallel to a generator of cone, the cutting profile is a parabola.



6. When a right circular cone is cut by a plane which is neither parallel to any generator / axis nor parallel to base, the cutting profile is an ellipse.



Hence a point, a pair of intersecting straight lines, circle, parabola, ellipse and hyperbola, all are conic-sections. All the conic sections are plane or two dimensional curves.

The conic section is the locus of a point which moves such that the ratio of its distance from a fixed point (focus) to the distance from a fixed straight line (directrix) is constant (e), e is called eccentricity of conic *i.e.*,

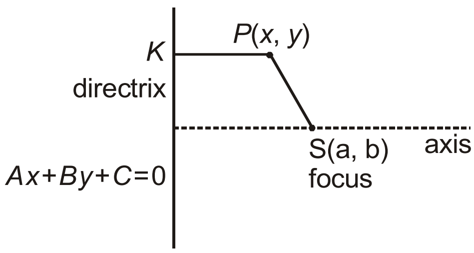
$$\frac{PS}{PK} = e$$

$e = 1$ for parabola

$e < 1$ for ellipse

$e > 1$ for hyperbola

A line through focus and perpendicular to directrix is called - axis. The vertex of conic is that point where the curve intersects the axis.



$$\frac{PS}{PK} = e \Rightarrow PS^2 = e^2 PK^2$$

or $(x-\alpha)^2+(y-\beta)^2=e^2\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2$

Simplification shall lead to the equation of the form

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

So equation of a conic is the general equation of second degree. If

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

if $\Delta = 0 \Rightarrow$ a pair of straight lines

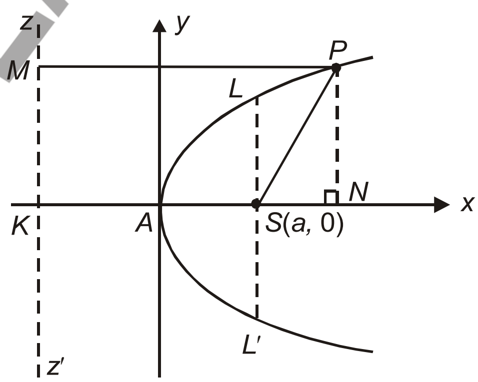
if $\Delta \neq 0$ and $h = 0$ and $a = b \Rightarrow$ a circle

if $\Delta \neq 0$ and $h^2 = ab \Rightarrow$ a parabola

if $\Delta \neq 0$ and $h^2 < ab \Rightarrow$ an ellipse

if $\Delta \neq 0$ and $h^2 > ab \Rightarrow$ a hyperbola

PARABOLA



The parabola shown has the equation $y^2 = 4ax$.

For this parabola

(i) Vertex A is (0, 0)

(ii) Focus S is $(a, 0)$

(iii) Equation of directrix zz' is $x = -a$.

(iv) Equation of axis AX is $y = 0$

Focal chord : A chord of a parabola which passes through the focus is called as focal chord.

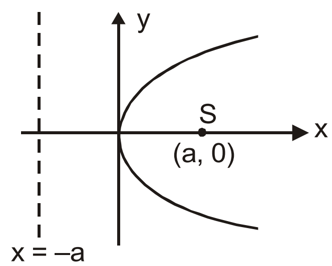
Focal distance : The distance, of a point on the parabola, from the focus is called as focal distance of the point.

Latus rectum : The double ordinate LSL' through the focus is the latus rectum. Its equation is $x = a$ and its length is $4a$ units.

Note : *(i) For parabola $PN^2 = 4AS.AN$.*

(ii) Two parabolas are said to be equal when their latus recta are equal.

Equations of Parabola in Standard Forms



(i) $y^2 = 4ax$

Focus : $(a, 0)$

Directrix : $x = -a$

Vertex : (0, 0)

Axis : $y = 0$

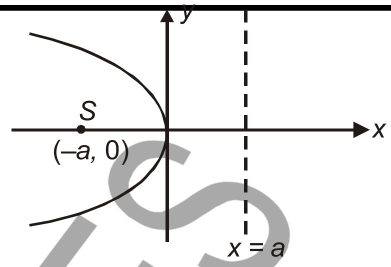
(ii) $y^2 = -4ax$

Focus : $(-a, 0)$

Directrix : $x = a$

Vertex : $(0, 0)$

Axis : $y = 0$



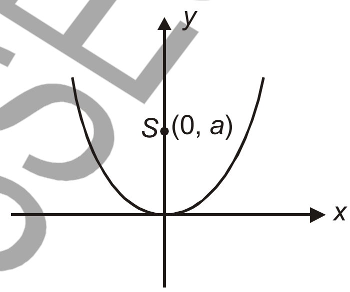
(iii) $x^2 = 4ay$

Focus : $(0, a)$

Directrix : $y = -a$

Vertex : $(0, 0)$

Axis : $x = 0$



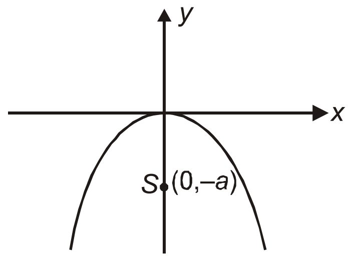
(iv) $x^2 = -4ay$

Focus : $(0, -a)$

Directrix : $y = a$

Vertex : $(0, 0)$

Axis : $x = 0$



Parametric co-ordinates

The parametric co-ordinates of any point on the parabola $y^2 = 4ax$ are given by :
$x = at^2$, $y = 2at$; t is the parameter

Position of a point with respect to the parabola $y^2 = 4ax$.

A point $P(x_1, y_1)$ lies inside, on or outside the parabola $y^2 = 4ax$ according as

$y_1^2 - 4ax_1 < 0, = 0$ or > 0

Equations of a Chord

Chord joining (x_1, y_1) and (x_2, y_2) on parabola $y^2 = 4ax$ is

$$y(y_1 + y_2) = 4ax + y_1y_2$$

Chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is
$y(t_1 + t_2) = 2(x + at_1t_2)$ — This chord will be a focal chord is $t_1t_2 = -1$

Length of chord $y = mx + c$ to the parabola $y^2 = 4ax$

$$\text{Length} = \frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)}$$

Length of focal chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2) = (t_2 - t_1)^2$

Length of focal chord through $(at^2, 2at)$ on $y^2 = 4ax = a\left(t+\frac{1}{t}\right)^2$

Chords and Tangents

Corresponding to any point $P(x_1, y_1)$ we define the following expressions for the conic

viz, $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$T \equiv axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c$

$S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

(i) Equation of tangent at $P(x_1, y_1)$ to $y^2 = 4ax$ is $T \equiv 0$
i.e. $yy_1 = 2a(x + x_1)$

(ii) Equation of tangent in slope form is $y = mx + \frac{a}{m}$

Its point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) The line $y = mx + c$ is a tangent to $y^2 = 4ax$ if $c = \frac{a}{m}$

(iv) If the line $y = mx + c$ intersects the parabola $y^2 = 4ax$ then, length of the chord intercepted is

$\sqrt{1+m^2}\ |x_2 - x_1|$

where x_1 and x_2 are abscissa of points of intersection and arise as roots of the equation

$$m^2x^2 + 2x(mc - 2a) + c^2 = 0$$

(i) Equation of pair of tangents

Through any given point $P(x_1, y_1)$ there pass, in general, two tangents to $y^2 = 4ax$. When P is external to the parabola, the joint equation to the pair of tangents from P is $SS_1 = T^2$

(ii) Points of intersection of two tangents

The tangents to the parabola $S \equiv y^2 - 4ax = 0$ drawn at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ intersect at the point $(at_1t_2, a(t_1 + t_2))$.

(iii) Chord of contact

Equation of chord of contact of the point $P(x_1, y_1)$ (i.e. of tangents from P) with respect to the parabola $y^2 = 4ax$ is $T \equiv 0$

(iv) Chord with middle point $P(x_1, y_1)$ has equation : $T \equiv S_1$

NORMALS TO THE PARABOLA

(i) Equation of normal to the parabola $y^2 = 4ax$ at the point $P(x_1, y_1)$ is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

(ii) Equation of normal to the parabola $y^2 = 4ax$ in slope form is $y = mx - 2am - am^3$
Its foot is given by $(am^2, -2am)$

(iii) Equation of normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y = -tx + 2at + at^3$

(iv) The normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ intersect at the point

$$(2a + a(t_1^2 + t_1t_2 + t_2^2), -at_1t_2(t_1 + t_2))$$

(v) Through a given point, in general, three normals can be drawn to a parabola

(vi) The sum of the slopes of the normals drawn from a given point to a parabola, is zero.

(vii) The sum of the ordinates of the feet of the normals drawn from a given point to a parabola is zero.

(viii) If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$

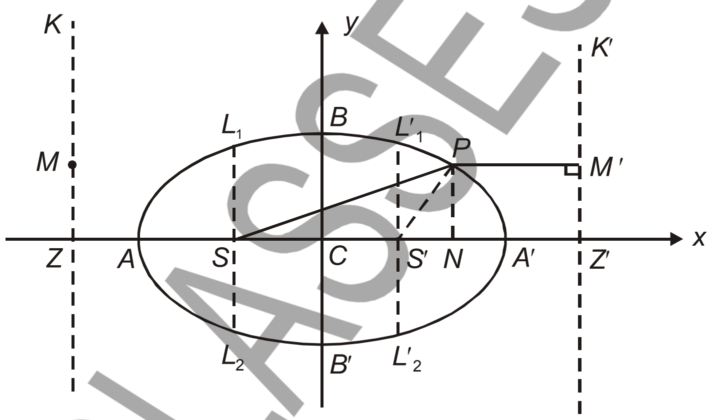
ELLIPSE

An ellipse is the locus of a point which moves such that its distance from a fixed point is e (< 1) times its distance from a fixed straight line. Unlike a parabola an ellipse has

(i) two foci
(ii) two directrices – one corresponding to each focus.
(iii) a center – the point such that all chords through it are bisected at it.
(iv) two axes – major and minor axes
(v) two vertices
(vi) two latus recta

Thus, the ellipse shown, has the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



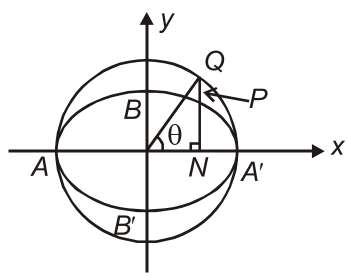
(i) Center is $C(0, 0)$
(ii) Vertices are $A(-a, 0)$ and $A'(a, 0)$
(iii) Foci are $S'(ae, 0)$ and $S(-ae, 0)$
(iv) Directrix $K'M'Z'$ corresponds to focus S' and has equation $x=\frac{a}{e}$.
Directrix KMZ corresponds to focus S and has equation $x=-\frac{a}{e}$
(v) L_1L_2 and $L_1'L_2'$ are double ordinates through the foci and are called latus rectum. Each has length $\frac{2b^2}{a}$.
(vi) AA' is the major axis and has length $2a$. BB' is the minor axis and has length $2b$ where $a > b$ and $b^2=a^2(1-e^2)$

Note : *1. An ellipse is the locus of a point which moves such that the sum of its distances from two fixed points remains constant. In fact, $PS + PS' = 2a$ for all points $P(x, y)$ on the ellipse.*

2. The constant in the above definition should be greater than the distance between the fixed points. This definition is called the physical definition of the ellipse.

Auxillary Circle

The circle which is described on the major axis AA' of an ellipse as diameter is called auxillary circle of the ellipse.



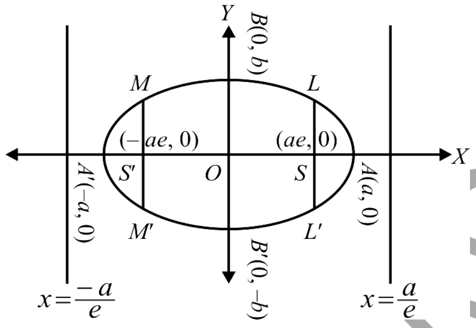
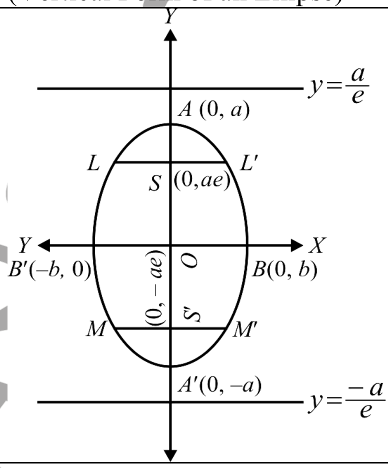
Equation of ellipse in parametric form

$$x = a\cos\theta,\ y = b\sin\theta\ ;\ 0 \le \theta < 2\pi$$

where θ is the parameter (as shown in the figure above) and is called as eccentric angle corresponding to the point P on the ellipse. The point Q has co-ordinates $(a\cos\theta, a\sin\theta)$. Further, $\frac{PN}{QN}=\frac{b}{a}$.

Position of a point with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The point $R(x', y')$ lies within, upon or outside the ellipse according as $\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}-1$ is negative, zero or positive.

Two Standard Forms of the Ellipse		
Standard Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ (Horizontal Form of an Ellipse)	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1\ (a, b)$ (Vertical Form of an Ellipse)
Shape of the Ellipse		
Centre	(0, 0)	(0, 0)
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{a}{e}$
Eccentricity	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{a^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Ends of latra-recta	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{b^2}{a}, \pm ae\right)$
Parametric coordinates	$(a\cos\theta, b\sin\theta)$	$(a\cos\theta, b\sin\theta)$
Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = a - ey_1$ and $S'P = a + ey_1$
Sum of focal radii $SP + S'P$ =	$2a$	$2a$
Distance between foci	$2ae$	$2ae$
Distance between directrices	$\frac{2a}{e}$	$\frac{2a}{e}$
Tangents at the vertices	$x = \pm a$	$y = \pm a$

CHORDS AND TANGENTS

Corresponding to any point $P(x_1, y_1)$ we define the following expressions for the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

viz., $$T \equiv axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c$$

$$S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

(i) The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at

two distinct points if $c^2 < a^2m^2 + b^2$

two coincident points if $c^2 = a^2m^2 + b^2$

imaginary points if $c^2 > a^2m^2 + b^2$

(ii) Length of the chord intercepted by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the line $y = mx + c$ is

$$\frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2}$$

(iii) Equation of tangent at the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$T \equiv 0 \; ; \; i.e., \; \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(iv) Equation of tangent at $(a \cos \theta, b \sin \theta)$ is $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$

(v) Equation of tangent in slope form is $y = mx \pm \sqrt{a^2m^2 + b^2}$

The point of contact of $y = mx + \sqrt{a^2m^2 + b^2}$ is $\left(\frac{-a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

and point of contact of $y = mx - \sqrt{a^2m^2 + b^2}$ is $\left(\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \frac{-b^2}{\sqrt{a^2m^2+b^2}}\right)$

(vi) **Point of intersection of two tangents**

Tangents at $(a \cos \theta_1, b \sin \theta_1)$ and $(a \cos \theta_2, b \sin \theta_2)$ intersect at

$$\frac{a \cos\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}, \; \frac{b \sin\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$$

(vii) **Equation of pair of tangents**

Through any given point $P(x_1, y_1)$ there pass, in general, two tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

When P is external to the ellipse, the joint equation of the pair of tangents from P is $SS_1 = T^2$

Chord of Contact

The equation of chord of contact of tangents drawn from $P(x_1, y_1)$ is $T \equiv 0$

Director Circle

It is the locus of point of intersection of tangents which are perpendicular to each other. Its equation is $x^2 + y^2 = a^2 + b^2$

Equation of Chord in Mid Point Form

Equation of the chord with middle point $P(x_1, y_1)$ is $T \equiv S_1$

Normals to the Ellipse

In general, four normals can be drawn from any point to an ellipse and the sum of the eccentric angles of their feet is equal to an odd multiple of two right angles

(i) Equation of normal at $P(x_1, y_1)$ is $\dfrac{x-x_1}{\frac{x_1}{a^2}} = \dfrac{y-y_1}{\frac{y_1}{b^2}}$

(ii) Equation of normal at $(a \cos \theta, b \sin \theta)$ is $ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$

(iii) Equation of normal in slope form is $y = mx \pm \dfrac{m(b^2 - a^2)}{\sqrt{b^2m^2 + a^2}}$

Equation of chord joining two points

The equation of chord joining the points $(a \cos \alpha, b \sin \alpha)$ and $(a \cos \beta, b \sin \beta)$ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

HYPERBOLA

The hyperbola is the locus of a point which moves such that its distance from a fixed point is e (> 1) times its distance from a fixed straight line. As in case of an ellipse, the hyperbola has

(i) two foci

(ii) two directrices – one corresponding to each focus.

(iii) a center – the point such that all chords through it are bisected there at.

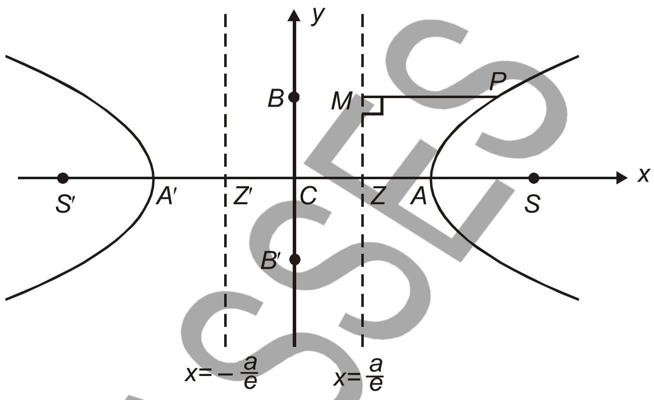
(iv) two axes – the transverse axis and conjugate axis.

(v) two vertices

(vi) two latus recta

Thus, the hyperbola shown in the adjacent figure has the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



(i) Centre is $C(0, 0)$

(ii) Vertices are $A'(-a, 0)$ and $A(a, 0)$

(iii) Foci are $S(ae, 0)$; $S'(-ae, 0)$

(iv) Directrices are $x=\pm\frac{a}{e}$

(v) AA' is the transverse axis and has length $2a$; BB' is the conjugate axes where B and B' are two points on the axis of y equidistant from C such that $CB = B'C = b$ and where $b^2=a^2(e^2-1)$.

(vi) Length of latus rectum is $\frac{2b^2}{a}$.

***Note :** The difference of the focal distances of any point on the hyperbola is equal to length of transverse axis ; thus,* $|PS-PS'|=2a$.

Equation of Hyperbola in Parametric Form

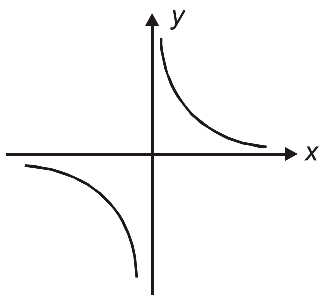
$x = a\sec\theta$; $y = b\tan\theta$, where $0 \le \theta < 2\pi$ (θ is the parameter)

Position of a Point with respect to Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The point $R(x', y')$ lies within, upon or outside the hyperbola according as $\frac{(x')^2}{a^2}-\frac{(y')^2}{b^2}-1$ is positive, zero or negative.

Equilateral or Rectangular Hyperbola

The hyperbola $x^2 - y^2 = a^2$, whose asymptotes are at right angles ($y = \pm x$) is called as an equilateral or rectangular hyperbola. If this hyperbola is rotated such that the coordinate axes coincides with the asymptotes, then, its equation reduces to $xy = c^2$ where $c^2=\frac{a^2}{2}$.



$xy = c^2$

(i) Parametric form of $xy = c^2$ is $x = ct$; $y=\frac{c}{t}$

(ii) Centre is (0, 0)

(iii) Transverse axis has equation $y = x$

Conjugate axis has equation $y = -x$

(iv) Its eccentricity is $e=\sqrt{2}$.

Tangent at $\left(ct, \frac{c}{t}\right)$ is $t^2 y + x = 2ct$

Normal at $\left(ct, \frac{c}{t}\right)$ is $xt^3 - ty - ct^4 + c = 0$

Chords and Tangents

Corresponding to any point $P(x_1, y_1)$, we define the following expressions for the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

viz., $T \equiv axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$

$$S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

(i) The line $y = mx + c$ intersects the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at

real and distinct point if $c^2 > a^2m^2 - b^2$

real and coincident points if $c^2 = a^2m^2 - b^2$

imaginary points if $c^2 < a^2m^2 - b^2$

(ii) Length of the chord intercepted by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ on the line $y = mx + c$ is

$$\frac{2ab\sqrt{b^2 + c^2 - a^2m^2}\sqrt{1 + m^2}}{\left|a^2m^2 - b^2\right|}$$

(iii) Equation of tangent at the point (x_1, y_1) lying on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T \equiv 0$;

i.e., $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(iv) Equation of tangent at $(a \sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

(v) Equation of tangent in slope form is $y = mx \pm \sqrt{a^2m^2 - b^2}$

The point of contact of $y = mx + \sqrt{a^2m^2 - b^2}$ is $\left(\frac{-a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{-b^2}{\sqrt{a^2m^2 - b^2}}\right)$

The point of contact of $y = mx - \sqrt{a^2m^2 - b^2}$ is $\left(\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Point of Intersection of Two Tangents

The tangents at $(a\sec\theta_1, b\tan\theta_1)$ and $(a\sec\theta_2, b\tan\theta_2)$ intersect at

$$\left(\frac{a\sin(\theta_2 - \theta_1)}{\sin\theta_2 - \sin\theta_1}, \frac{b(\cos\theta_1 - \cos\theta_2)}{\sin\theta_2 - \sin\theta_1}\right)$$

Equation to Pair of Tangents

Through any given point $P(x_1, y_1)$ there pass in general, two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

When P is external to the hyperbola, the joint equation to the pair of tangents from P is $SS_1 = T^2$

i.e., $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$

Chord of Contact

Equation of chord of contact of the point (x_1, y_1) with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T \equiv 0$

Director Circle

It is the locus of point of intersection of tangents which are perpendicular to each other. Its equation is $x^2 + y^2 = a^2 - b^2$

Equation of Chord in Mid Point Form

Equation of the chord with middle point (x_1, y_1) is $T \equiv S_1$

Equation of Normal

(i) Equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$

(ii) Equation of normal at $(a \sec\theta, b \tan\theta)$ is $a x \cos\theta + b y \cot\theta = a^2 + b^2$