11
CHAPTER

# Circle

**AIEEE Syllabus :** Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equation of a circle in the parametric form, equation of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to the circle, length of the tangent, equation of the tangent, equation of a family of circles through the intersection of two circles, condition for two intersecting circles to be othogonal.

*Among all curves, circle is the simplest curve. It is common in almost every sphere of life. A wheel, a circular object, has revolutionised the transportation like all motor vehicles, railways. pullies, gears, various rings are all circular.*

## CIRCLE STANDARD EQUATION

A circle is the locus of a point which moves such that its distance from a fixed point, called as centre, is equal to a given distance, called as radius. Thus, equation to a circle with centre (α, β) and radius $a$ is : $(x-\alpha)^2+(y-\beta)^2=a^2$

If the centre is origin then $x^2 + y^2 = a^2$

### General Equation of Circle

The general equation of a circle is :

$$S \equiv x^2+y^2+2gx+2fy+c=0$$

It has three arbitrary constants. Its centre is $(-g, -f)$ and radius is $\sqrt{g^2+f^2-c}$ . The circle is real, point circle or imaginary according as $g^2+f^2-c>0, =0,$ or $<0$; $c=0$ if circle passes through origin.

### General Equation of Second Degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ shall represent a circle if

Coefficient of $x^2$ & $y^2$ are equal i.e. $a = b$

& Coefficient of $xy$ is zero i.e. $h = 0$

## DIFFERENT FORMS OF EQUATIONS OF CIRCLES

### (a) Parametric Form

Circle with centre (α, β) and radius $a$ is $(x - \alpha)^2 + (y - \beta)^2 = a^2$

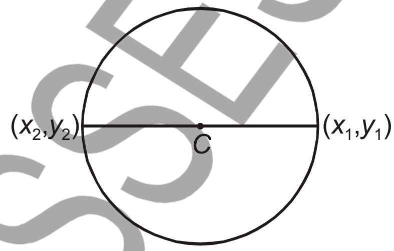
It can be represented in parametric form as

$x = \alpha + a\cos\theta$, $y = \beta + a\sin\theta$ — $\theta$ is parameter, $0 \le \theta \le 2\pi$

A circle with centre (0, 0) and radius $a$ is

$x = a\cos\theta$, $y = a\sin\theta$ — $\theta$ is parameter

**(b) Equation of Circle in Diametric Form**




If two diametrically opposite points on a circle are $(x_1, y_1)$ and $(x_2, y_2)$ then centre $= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Radius $= \frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

and equation of circle is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

**Intercepts Made by a Circle on Axes**

(i) Intercept made on $x$-axis by $S \equiv 0$ is : $2\sqrt{g^2-c}$

(ii) Intercept made on $y$-axis by $S \equiv 0$ is : $2\sqrt{f^2-c}$

(iii) If $g^2 = c$ (resp., $f^2 = c$), the circle touches the $x$-axis (resp. $y$-axis).
if $c = g^2 = f^2$ then circle touches both axes.

(iv) Length of the chord intercepted by $x^2+y^2=a^2$ on the line $y = mx + c$ is $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$

(v) Intercepts are always positive.

**Position of a Point with respect to Circle**

A point $P(x_1, y_1)$ lies inside, on or outside the circle $S \equiv x^2+y^2+2gx+2fy+c=0$ according as $x_1^2+y_1^2+2gx_1+2fy_1+c$ is $< 0$, $= 0$ or $> 0$.

**TANGENT TO A CIRCLE**

Corresponding to any point $P(x_1, y_1)$, we define the expressions

$$S_1 \equiv x_1^2+y_1^2+2gx_1+2fy_1+c$$

$$T \equiv xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$$

(i) Equation of **tangent at $(x_1, y_1)$** to $S \equiv 0$ is : $T = 0$

(ii) Equation of **normal at $(x_1, y_1)$** to $S \equiv 0$ is $y(x_1+g)-x(y_1+f)+fx_1-gy_1=0$

(iii) The line $y = mx + c$ intersects the circle $x^2+y^2=a^2$ at
real and distinct points if $c^2 < a^2(1 + m^2)$
real and coincident points if $c^2 = a^2(1 + m^2)$
imaginary points if $c^2 > a^2(1 + m^2)$
$\therefore$ $y = mx + c$ is a tangent to $x^2+y^2=a^2$ if $c^2=a^2(1+m^2)$

(iv) The line $y = mx + a\sqrt{1+m^2}$ is a tangent to $x^2 + y^2 = a^2$ at $\left(\dfrac{-am}{\sqrt{1+m^2}}, \dfrac{a}{\sqrt{1+m^2}}\right)$

The line $y = mx - a\sqrt{1+m^2}$ is tangent to $x^2 + y^2 = a^2$ at $\left(\dfrac{am}{\sqrt{1+m^2}}, \dfrac{-a}{\sqrt{1+m^2}}\right)$.

(v) Length of the tangent from $P(x_1, y_1)$ to $S \equiv 0$ is equal to $\sqrt{S_1}$, where $P$ lies outside $S \equiv 0$.

(vi) Equation to the pair of tangents from $P(x_1, y_1)$ to $S \equiv 0$ is given by $SS_1 = T^2$.

### Angle of Intersection of Two Circles

(i) Angle of intersection of two circles $S_1 = 0$ and $S_2 = 0$

Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + C_1$

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + C_2$

Let $S_1$ and $S_2$ have centres $C_1$ and $C_2$ respectively and radii $r_1$ and $r_2$ respectively. The angle of intersection $\theta$ of the circles is the angle between their tangents at points of intersection and is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - (C_1C_2)^2}{2r_1r_2}$$

(ii) For Orthogonal circles :

Circles $S_1 \equiv 0$ and $S_2 \equiv 0$ are said to intersect orthogonally if $\theta = 90°$ ;

*i.e.*, if $2g_1g_2 + 2f_1f_2 = C_1 + C_2$

### Equation of a Circle through Intersection Points of Circle and Line

Let the circle be $S = 0$ and line be $L = 0$

$S + \lambda L = 0$ is a circle which passes through the points of intersection of both.

(i) Equation of any circle that passes through two given points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Let $u \equiv ax + by + k = 0$

(ii) If $u \equiv 0$ is a tangent to $S \equiv 0$ at the point $P$ then $S + \lambda u \equiv 0$ is the equation of circles touching $S \equiv 0$ at $P$.

(iii) Equation of the circles which touch the line $u \equiv 0$ at $(x_1, y_1)$ is $(x - x_1)^2 + (y - y_1)^2 + \lambda u \equiv 0$

### Equation of a Circle through Intersection of Two Circles

(i) If $S_1 \equiv 0$ and $S_2 \equiv 0$ be two circle, intersecting in real points, then $S_1 + \lambda S_2 = 0$ $(\lambda \neq -1)$ is the equation of the family of circles passing through the common points of $S_1 \equiv 0$ and $S_2 \equiv 0$.

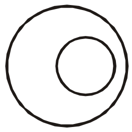
(ii) If $S_1 \equiv 0$ and $S_2 \equiv 0$ intersect, then $S_1 - S_2 \equiv 0$ is the equation of their common chord.

(iii) If $S_1 \equiv 0$ and $S_2 \equiv 0$ touch each other then $S_1 - S_2 \equiv 0$ is the equation of their common tangent at the point of contact.

## Common Tangents to Two Circles

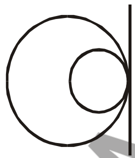
Let circles $S_1 \equiv 0$ and $S_2 \equiv 0$ have centers $O_1$ and $O_2$ respectively and radii $r_1$ and $r_2$ respectively. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is

(i) **zero,** if one circle lies totally inside the other



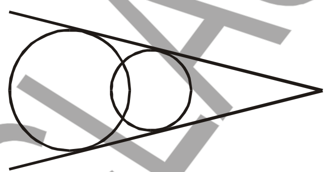
$O_1O_2 < r_1 - r_2$

(ii) **one,** if circles touch each other internally



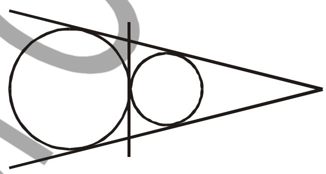
$O_1O_2 = |r_1 - r_2|$

(iii) **two,** if circles intersect at two distinct points



$|r_1 - r_2| < O_1O_2 < r_1 + r_2$

(iv) **three,** if circles touch each other externally

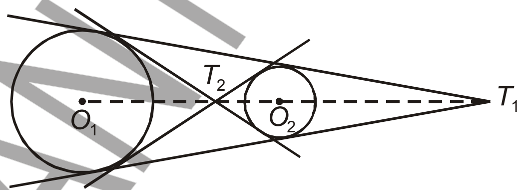


$O_1O_2 = r_1 + r_2$

(v) **four,** if circles lie outside each other

Two of the tangents are **direct common tangents.**

Two of the tangents are **transverse common tangents.**




$O_1O_2 > r_1 + r_2$

If direct common tangents meet the line $O_1O_2$ in $T_1$ then $T_1$ divides the segment $O_1O_2$ externally in the ratio $r_1 : r_2$.

If the transverse common tangents meet the line $O_1O_2$ in $T_2$ then $T_2$ divides the segment $O_1O_2$ internally in the ratio $r_1 : r_2$.

The coordinates of $T_1$ and $T_2$ having been found, the corresponding tangents are straight lines through it such that perpendiculars on them from $O_1$ are each equal to $r_1$.

> ***Note :*** *If $S_1 = 0$ and $S_2 = 0$ touch each other, then their point of contact can be obtained by solving $S_1 = 0$ and $S_1 - S_2 = 0$ simultaneously.*

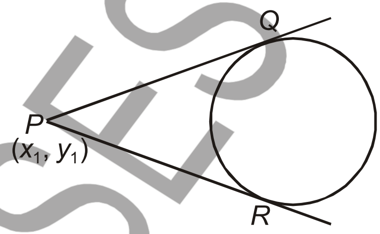
## Radical Axis, Chord of Contact and Chord with the given Middle Point

The radical axis of $S_1 \equiv 0$ and $S_2 \equiv 0$ is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

The equation to the radical axes is : $S_1 - S_2 = 0$.

**1. Equation of the pair of tangents :**

Combined equation of the tangents $PQ$ and $PR$ drawn from $P(x_1, y_1)$ to the circle $S = 0$ is given by




$\boxed{SS_1 = T^2}$, where

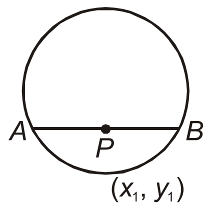
$$S = x^2 + y^2 + 2gx + 2fy + c$$

$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

**2. Equation of the chord whose middle point is given :**

If $P(x_1, y_1)$ be the middle point of a chord $AB$ of the circle $S = 0$, then equation of the chord is given by



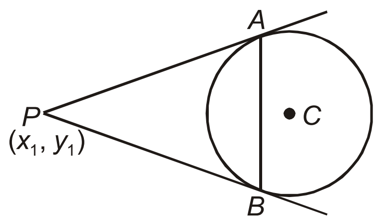

$$\boxed{T = S_1}$$

*i.e.*, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$

**3. Chord of contact**

It is the line joining the points of contact of the tangents drawn from a point outside the circle.

If from a point $P(x_1, y_1)$, two tangents $PA$ and $PB$ are drawn. The line $AB$ is the chord of contact of the point $P$ with respect to the given circle.



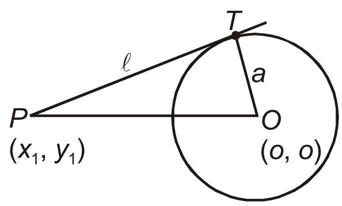

Equation of the chord of contact with respect to the

(i) Circles $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

(ii) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

**4. Length of the tangent from a point to the circle**

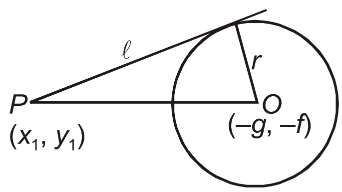
Let $P(x_1, y_1)$ be a point and $PT$ is the tangent drawn from $P$ to the circle $S \equiv x^2 + y^2 - a^2 = 0$, then




$$PT = \sqrt{OP^2 - OT^2}$$

$$= \sqrt{x_1^2 + y_1^2 - a^2}$$

If the circle is $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then




$$PT = \sqrt{OP^2 - OT^2}$$

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$