10
CHAPTER

The Straight Line

AIEEE Syllabus : Recall of Cartesian system of rectangular co-ordinates in a plane, distance formula, area of a triangle, condition for collinearity of 3 points and section formula. Centroid & incentre of a triangle, locus and its equation, translation of axes, slope of a line, parallel and perpendicular lines, intercepts of a line on the co-ordinate axes.

Coordinate geometry is the study of geometry using algebra. The plane curves which would be considered are : straight line, circle, parabola, ellipse and hyperbola.

The system of coordinates used here is cartesian system of coordinates, introduced by philosopher Descartes. It is by far the most important system.

DISTANCE FORMULA

Distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ *i.e.* length of line PQ is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

AREA OF TRIANGLE

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ form a triangle then area of ΔABC is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right| \text{ 'square unit'}$$

Condition for Collinearity of 3-points

If A, B & C are collinear, the area of triangle ABC has to be zero *i.e.*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or } x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

AREA OF A PLANE POLYGON

Let $A_1, A_2 \ldots\ldots\ldots A_n$ are the vertices of a n sided plane polygon, then

$$\text{Area of Polygon } = \frac{1}{2}\left|[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots\ldots\ldots + (x_ny_1 - x_1y_n)]\right| \text{ sq. unit}$$

Important results

To prove that a given four sided figure is :

1. **Square** : Prove that four sides are equal & diagonals are equal
2. **Rhombus** : Four sides equal, diagonals unequal.

3. Rectangleyes : Opposite sides equal & diagonals equal.

4. Parallelogram : Opposite sides equal & diagonals unequal.

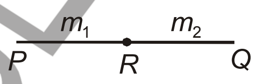
Note : *Diagonals bisect each other in all the above.*

SECTION FORMULA

Coordinates of the point $R(\bar{x}, \bar{y})$ which divides the join of the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

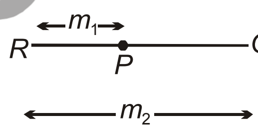
(i) internally, in the ratio $m_1 : m_2$, are

$$\bar{x} = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \;;\; \bar{y} = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$



(ii) externally, in the ratio $m_1 : m_2$, are

$$\bar{x} = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \;;\; \bar{y} = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$



where $m_1 \neq m_2$

Cor 1 : Mid point ($m_1 = m_2$) of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Cor 2 : For finding the ratio of division, use $\lambda : 1$. If λ comes out to be positive, it indicates internal division otherwise external if λ is negative.

Cor 3 : Line $Ax + By + C = 0$ divides join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\left[-\frac{(Ax_1 + By_1 + C)}{(Ax_2 + By_2 + C)}\right]$

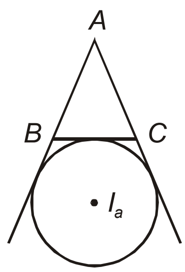
CENTROID, INCENTRE & EX-CENTRES OF A TRIANGLE

If vertices of ΔABC have coordinates $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

(i) Coordinates of its *centroid* are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

(ii) Coordinates of its *incentre* are $\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$

where $a = BC$, $b = AC$ and $c = AB$



(iii) Coordinates of *excentre* I_a are

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

If G, C and H denote the centroid, circumcentre and orthocentre respectively of ΔABC, then, G, C and H are collinear and G divides CH internally in the ratio 1 : 2.

Note : *Incentre is the point of intersection of internal bisectors of angles of triangle. Its distance from all three sides is same and called inradius (r) of circle.*

Circumcentre of Triangle

It is the point of intersection of perpendicular bisectors of sides, so its distance from all three vertices is same.

Circumcentre is given by

$$\left[\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right]$$

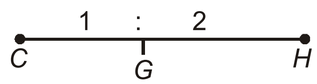
Orthocentre

It is the point of intersection of the perpendicular drawn from the vertices of the triangle on the opposite sides. When vertices and the angles of the triangle are given, then orthocentre is given by,

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

Note :

1. *In case of equilateral triangle, centroid, incentre, circumcentre and orthocentre of the triangle lie at the same point.*
2. *Orthocentre H, centroid G and circumcentre C of a triangle are collinear and centroid divides the line joining orthocentre and circumcentre in the ratio of 1 : 2*



LOCUS AND ITS EQUATION

When a point moves, so as always to satisfy a given condition or conditions, the path it traces out is called as its locus under these conditions.

Standard method for writing equation of a locus

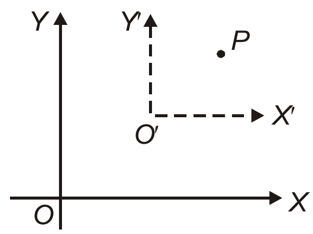
1. Assume the point whose locus (path) is to be found is (h, k).
2. Make the equation involving (h, k) as per the conditions given.
3. Simplify this equation.
4. Replance h by x and k by y in the simplified form of equation & you get the equation of locus.

Shifting of Origin

Change of axes, by changing origin, the direction of axes remaining the same.

Let OXY and $O'X'Y'$ be two rectangular Cartesian system of axes.

Let P be any point in the plane of the axes and let P and O' have coordinates (x, y) and (h, k) respectively with respect to OXY system. Then the coordinates (x', y') of P with respect to the system $OX'Y'$ are given by $x = x' + h$; $y = y' + k$



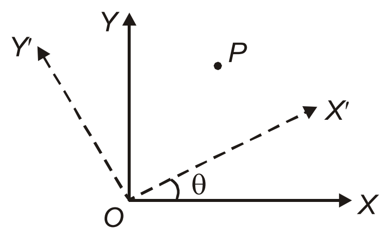
Rotation of axes

Change of axes (without changing the origin), by changing the direction of axes, both systems of coordinates being rectangular.

If a point P in the plane of OXY has coordinates (x, y) and (x', y') with respect to the system OXY and $OX'Y'$ respectively, then

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$



SLOPE OF A LINE

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points then slope between A and B is defined as :

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta \quad : \quad 0 \le \theta < \pi \,; \quad \theta \ne \frac{\pi}{2}$$

where θ is the angle of inclination of the line joining A and B with the positive direction of x-axis. If $x_1 = x_2$ then slope is not defined.

Intercepts of a Line

Let a line $L \equiv ax + by + c = 0$, intersects OX - axes at A and OY - axes at B, then OA and OB are called x-intercept and y-intercept of line respectively. For x - intercept, substitute $y = 0$ in the equation

i.e. $ax + c = 0$

$\therefore \quad x = -c/a$ is the x - intercept

Similarly for y - intercept put $x = 0$

$by + c = 0$

$\therefore \quad y = -c/b$ is the y - intercept

Note : *The equation of line having its x and y-intercepts as 'a' and 'b' respectively is*

$\frac{x}{a} + \frac{y}{b} = 1$ *(called intercept form of line)*

GENERAL EQUATION OF A STRAIGHT LINE

An equation of the form $ax + by + c = 0$ where a & b both are not zero simultaneously, represents a straight line.

whose slope is $-\frac{a}{b}$

Note :

(1) Equation (i) x-axis is $y = 0$

(ii) y-axis is $x = 0$

(2) Equation of line (a is a non-zero constant)

(i) parallel to x-axis is $y = a$

(ii) parallel to y-axis is $x = a$

VARIOUS FORMS OF EQUATION OF STRAIGHT LINE

(1) Line with slope m and a given point (x_1, y_1) on it

$y - y_1 = m(x - x_1)$ **(Slope point form)**

(2) Line with two given points (x_1, y_1) and (x_2, y_2) on it

$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$ **(Two point form)**

(3) Line with given slope m and intercept c on y-axis

$y = mx + c$ **(Slope intercept form)**

(4) Line with given intercepts a and b on x and y axes respectively

$\frac{x}{a} + \frac{y}{b} = 1$ **(Double intercept form)**

(5) Line at perpendicular distance p from the origin and where, the perpendicular makes angle α with OX

$x \cos \alpha + y \sin \alpha = p$ **(Normal or perpendicular form)**

(6) Line making an angle α with OX and passing through (x_1, y_1) and r as the directed distance of any point $P(x, y)$ on the line

$x = x_1 \pm r \cos \alpha \;;\; y = y_1 \pm r \sin \alpha$ **(Symmetric or parametric form)**

Angle between Two Lines

$\tan\phi = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$ where ϕ is acute angle and m_1 & m_2 are slopes of two lines

Otherwise $\tan\phi = \pm\left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$; one sign gives acute angle and the other gives obtuse angle

Here m_1 or m_2 or both should not be infinite.

If both m_1 & m_2 are undefined, angle = 0

If m_1 is undefined & $m_2 = \tan\theta$, then angle between lines is $\left(\dfrac{\pi}{2} - \theta\right)$

For parallel lines $m_1 = m_2$ and for perpendicular lines $m_1 m_2 = -1$

Points in Relation to a Line $ax + by + c = 0$

1. The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side or on the opposite side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have same sign or opposite sign.
2. The ratio in which the line $ax + by + c = 0$ divides the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $-\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$.
3. The length of the perpendicular from $P(x_1, y_1)$ to $ax + by + c = 0$ is $P = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$
4. Coordinates of the foot of the perpendicular drawn from $P(x_1, y_1)$ to the line $ax + by + c = 0$ are given by $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = -\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$
5. Coordinates of the image of $P(x_1, y_1)$ in the line $ax + by + c = 0$ are given by
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{\sqrt{a^2 + b^2}}$$
6. Line parallel to ax + by + c = 0 is ax + by + k = 0 adn line perpendicular to ax + by + c = 0 is bx – ay + k = 0.

CONCURRENCY OF LINES

The lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if they pass through the same point ; the condition for concurrency is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Note : *To prove three line to be concurrent show that the point of intersection of any two lines, satisfies the equation of third line.*

FAMILY OF LINES

Lines through the point of intersection of two given lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by : $(a_1x + b_1y + c_1) + \lambda\,(a_2x + b_2y + c_2) = 0$ where λ is a parameter.

It represents a family of lines. Any particular line (member of the family) can be found from the additional condition stated about the required line.

ANGLE BISECTORS BETWEEN TWO LINES

The equations of the bisectors of the angles between two intersecting lines

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

Let ϕ be the angle between one of the bisectors and one of the lines $a_1x + b_1y + c_1 = 0$. If $|\tan\phi| < 1$ i.e. $\phi < 45°$, then that bisector is the acute angle bisector of the two given lines. The other equation represents the obtuse angle bisector.

Rule for writing a particular bisector

Write the equations of the lines so that constant terms are positive

If $a_1a_2 + b_1b_2 > 0$ then $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$ …(1)

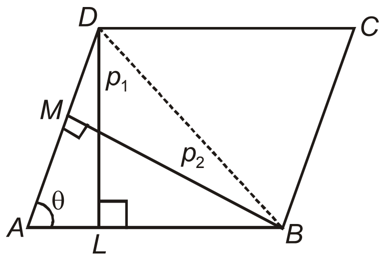
gives the obtuse angle bisector. If $a_1a_2 + b_1b_2 < 0$ then (1) gives the acute angle bisector.

Note : *If the equations are written so that the constant terms have same sign, then (1) gives bisector of the angle in which the origin lies (and not necessarily the acute angle bisector)*

AREA OF PARALLELOGRAM OR RHOMBUS

Area of a parallelogram or a rhombus, equations of whose sides are given, can be obtained by using the following formula

$$Area = \frac{p_1p_2}{\sin\theta}$$



Where $p_1 = DL$ = distance between lines AB and CD,
$p_2 = BM$ = distance between lines AD and BC,
θ = angle between adjacent sides AB and AD.

In the case of a rhombus, $p_1 = p_2$. Thus, Area of rhombus = $\dfrac{p_1^2}{\sin\theta}$

Also, area of rhombus = $\dfrac{1}{2}d_1d_2$

where d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus.