1 CHAPTER

Sets, Relations and Functions

AIEEE Syllabus : Sets and their representation, Union, Intersection and Complements of sets and their Algebraic properties, Relations, Equivalence relations, Mapping, one-one, Into and Onto mappings, Composition of mappings

*The concept of **set** is the basis of the modern Mathematics. It is widely used in various branches of Mathematics.*

*The concept of **relation** is very useful to understand a function. If function or not as a function can only be understood the concept of relation is clear.*

'aRb' means 'a' is R-related to b', where R may be any given relation between a & b.

*The concept of **function** lays the foundation of the study of the most important branch 'calculus' of mathematics. The word 'function' is derived from a Latin word meaning 'operation'. Function is also called mapping.*

SET - "Set is a well-defined collection of distinct objects"

The objects of a set have a common property. An object having this property belongs to this set and another object not possessing this property does not belong to that set.

For example, the collection of books written by Shakespere is a set, but the collection of interesting books written by Shakespere is not a set, since a book found interesting by one person may not be liked by another.

Representation of a Set

A set can be represented by two methods

1. Roster form or tabular form
2. Set builder form or rule method.

Roster or Tabular Form

Here the elements of set are listed seperated by commas within braces or curly brackets {}. Here order of listing is immaterial and no element is repeated.

For example, the set A of all single digit natural numbers is written as

A = {1, 2, 3, 4, 5, 6, 7, 8, 9} or A = {1, 3, 5, 2, 6, 4, 9, 8, 7}

(order is immaterial)

Set-Builder Form:

Here we choose a variable (say x), which represents each element of the set satisfying a particular property. Inside the bracket, x is followed by symbol: (or ; or vertical line '|' or oblique line '/' followed by the property or properties, possessed by each element of set. For example, the set A of all even integers less than 10 is written as

A $= \{x : x$ is an even integer less than $10\}$

$= \{x \mid x$ is an even integer less than $10\}$

$= \{x ; x$ is an even integer less than $10\}$

$= \{x / x$ is an even integer less than $10\}$

The symbol following x is read as 'such that'.

The roster form of A is written as

A = {0, 2, 4, 6, 8}

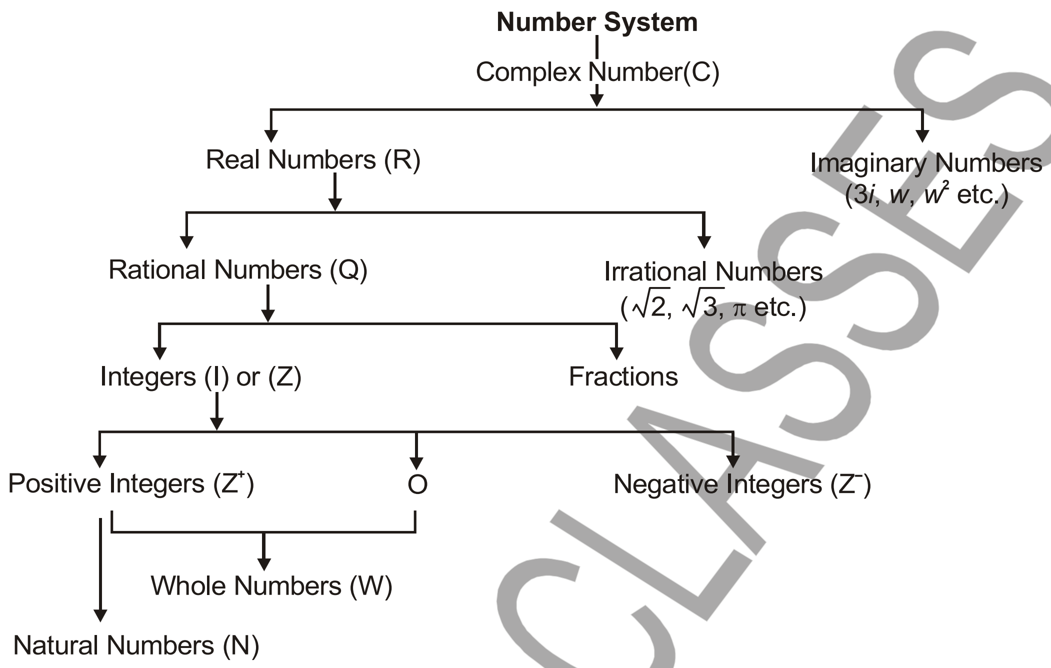
Note : '0' is an even integer

Set builder form is also called, rule method, property method or symbolic method.

Standard Notations for Sets of Numbers

	Set of all	Symbol	*i.e.*
1.	Natural number	N	N = {1, 2, 3.......}
2.	Integers	Z or I	Z or I = {.......–3, –2, –1, 0, 1, 2, 3,....}
3.	(a) Positive integers	Z^+	$Z^+ = \{1, 2, 3,......\}$
	(b) Negative integers	Z^-	$Z^- = \{.....-3, -2, -1\}$
4.	Integers excluding 0	I_0	$I_0 = \{\pm1, \pm2, \pm3........\}$
5.	Even integers	E	$E = \{0, \pm2, \pm4........\}$
6.	Odd integers	O	$O = \{\pm1, \pm3, \pm5........\}$
7.	Rational numbers	Q	$Q = \{x : x = \frac{p}{q}, p$ and q are integers, $q \neq 0\}$
8.	Non-zero rational numbers	Q_0	$Q_0 = \{x : x \in Q, x \neq 0\}$
9.	Positive rational numbers	Q^+	$Q^+ = \{x : x \in Q, x > 0\}$
10.	Real numbers	R	Here all rational and irrational numbers are included
11.	Non-zero real numbers	R_0	$R_0 = \{x : x \in R, x \neq 0\}$
12.	Positive real number	R^+	$R^+ = \{x : x \in R, x > 0\}$
13.	Complex numbers	C	$C = \{a + ib; a, b \in R$ and $i = \sqrt{-1}\}$
14.	Non-zero complex number	C_0	$C_0 = \{x : x \in C, x \neq 0\}$
15.	Natural numbers less than or equal to K, where K is positive integer	N_k	$N_k = \{1, 2, 3, 4, ...k\}$
16.	Whole numbers	W	W = {0, 1, 2, 3,......}

NUMBER SYSTEM



Types of Sets

1. Null Set (or Empty Set or Void Set)

A set which has no element. It is denoted by ϕ or $\{\}$.

2. Singleton Set

A set having a single element only *e.g.* $\{\phi\}$, $\{0\}$, $\{2\}$, $\{a\}$ etc. each is sigleton set or unit set.

3. Pair–Set

A set having two elements only.

e.g. $\{0, 1\}$, $\{\pm 1\}$, $\{x : x$ is a root of $x^2 - 5x + 6 = 0\}$

4. Set of Sets

A set S having all its elements as sets is called set of sets or a family of sets or a class of sets.

e.g. $\{\{1, 2\}, \{2, 3\}\ \{1, 2, 3\}\}$ is a set of sets as each member is a set itself.

$\{\{1, 2\}, 7, \{1, 7, 4\}\}$ is not a set of set as 7 is not a set.

5. Finite and Infinite Set

A set having finite number of elements in it is called finite set otherwise infinite set. The number of members in an infinite set are infinite *i.e.* can not be counted. The number of elements in a finite set A is called **Cardinal number, n(A)**, of set A.

6. Equivalent Sets :

Two sets A and B are equivalent iff $n(A) = n(B)$.

e.g. $A = \{a, b, c, d, e\}$ and $B = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$ are equivalent sets as $n(A) = n(B) = 5$

7. Equal Sets:

Two sets A and B are equal if both have all the elements same *i.e.* A is a subset of B and B is also a subset of A. The order of elements is immaterial.

$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

e.g. $\{a, b, c\} = \{a, c, b\}$

$\boxed{A = B \text{ if } x \in A \Rightarrow x \in B \text{ and } x \in B \Rightarrow x \in A}$

SUBSETS, SUPERSETS, PROPER SUBSETS

A set 'A' is called a subset of set B if every member of set A also belongs to set B. (sign of subset is $\subseteq$)

$A \subseteq B \Leftrightarrow [x \in A \Rightarrow x \in B] \Rightarrow$ (A is contained in B)

Here set B is called superset of A. $B \supseteq A$

$A \not\subseteq B$ is read as 'A is not a subset of B'.

Example :

$A = \{x : x \in N\}$ and $B = \{x : x \in Z\}$

So $A \subseteq B$ as every natural number is also an integer.

Example :

$A = \{a, e, i, o, u\}$; $B = \{x : x$ is a letter of English alphabet$\}$

$\therefore \quad A \subseteq B$

Example :

$A = \{a, e, i, o, u\}$ and $B = \{e, o, u, i, a\}$

$A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

Proper Subset

A set A is said to be a proper subset of a set B if every element of Set A is an element of set B and set B has atleast one extra element which is not an element of A.

Proper subset is denoted by $A \subset B$ (read as "A is a proper subset of B")

$A \not\subset B$ (read as "A is not a proper subset of B")

Comparability of Sets

Two sets A and B are called comparable if $A \subset B$ or $B \subset A$ or $A = B$, otherwise A and B are called incomparable.

Example :

$\{a, e, i\}$ and $\{a, e, o\}$ are incomparable.

$\{a, e, i\}$ and $\{a, e, i, o, u\}$ are comparable.

$\{a, e, i\}$ and $\{e, i, a\}$ are comparable.

Power Set

The set of all subsets of a set A is called power set of A and is denoted by P(A) or 2^A.

$P(A) = \{x : x \subseteq A\}$

$x \in P(A) \Leftrightarrow x \subseteq A$

$\phi \in P(A)$ and $A \in P(A)$

Example :

A = {1, 2, 3}

P(A) = 2^A = {ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

n(A) = 3 so n(2^A) = 2^3 = 8

(i) If A has n elements then its power set P(A) contains 2^n elements. nP(A) = 2^n.

(ii) ϕ and A both belong to P(A).

(iii) If A = ϕ then P(A) = {ϕ} is a singleton set.

(iv) If A = {*t*} then P(A) {ϕ, {*t*}} is a pair set.

(v) If cardinal number of set A is n then total number of subsets of P(A) = 2^{2^n} and proper subsets = $2^{2^n} - 1$.

(vi) If A ⊆ B ⇒ P(A) ⊆ P(B)

Universal Set

Any set which is the superset of all the sets under consideration is called the universal set (Ω or S or U).

Choice of universal set is not unique, but once chosen it is fixed for that discussion.

Example:

Let A = {a, e, i}; B = {i, o, u}; C = {e, f}

then U = {a, e, i, o, u, f}

or U = {a, e, i, p, o, u, f, g}

or U = Set of all English alphabet.

Intervals as Subsets of R

Four type of subsets can be defined on R as given below.

Let a, b ∈ R, such that $a < b$

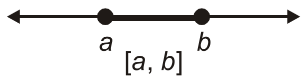
1. Open Interval

] a, b[or (a, b)

(a, b) or] a, b [= {x : $a < x < b$}

= Set of all real numbers between a and b, not including a and b both.

2. Closed Interval



[a, b] = {x : $a \le x \le b$}

= Set of all real numbers between a and b as well as including a and b both.

3. Open-closed Interval (semi closed or semi open interval)

(a, b] or]a, b] = {x : $a < x \le b$}

= Set of all real numbers between a and b, a not included but b included.

(a, b] or]a, b]

4. Closed-open interval (semi closed or semi open interval)

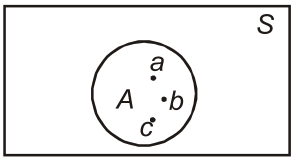
[a, b) or [a, b[

[a, b) or [a, b[= {x : $a \le x < b$}

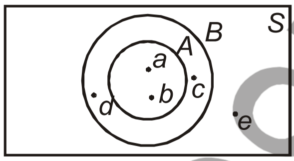
= Set of all real numbers between a and b including a but excluding b.

VENN DIAGRAMS

Introduced by Euler (a Swiss mathematician) and named after John Venn. It is a pictorial representation of sets in which a set is represented by a circle or a closed geometrical figure inside universal set which is shown by a rectangle. Each element of a set is represented by a point within the circle representing that set.



$A = \{a, b, c\}$

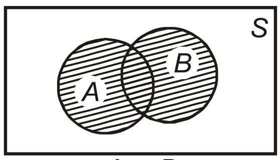


$A \subset B$; $A = \{a, b\}$; $B = \{a, b, c, d\}$; $e \notin A$; $e \notin B$

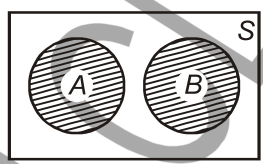
Various Operations on Sets

Union of Sets : $A \cup B$ (read as 'A union B' or 'A cup B' or 'A join B') is a set consisting of all the elements which are either in A or in B or in both.

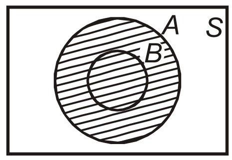
$A \cup B = \{x : x \in A, x \in B\}$



$A \cup B$



$A \cup B$



$A \cup B$

$x \in A \cup B \Leftrightarrow x \in A \lor x \in B$

$x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$

'$\lor$' denotes 'or'

INTERSECTION OF TWO SETS

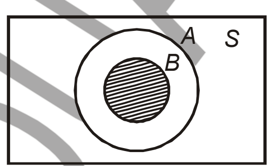
$A \cap B$ (read as 'A intersection B' or 'A cap B' or 'A meet B') is defined as a set containing all the elements common to A and B.

$A \cap B = \{x : x \in A$ and $x \in B\}$

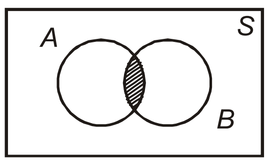
$\quad = \{x : x \in A \land x \in B\}$

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

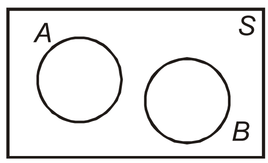
$x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$



$A \cap B$



$A \cap B$



$A \cap B = \phi$

$A_1 \cap A_2 \cap A_2$ .... $\cap A_n = \bigcap_{i=1}^{n} A_i = \{x : x \in A_i \ \forall\ i\}$ $A \cap B = AB$

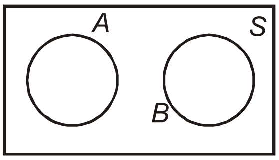
Intersection of finite number of finite sets will be a finite set
Intersection of finite set with infinite set will be finite set
Intersection of two or more infinite sets may or may not be finite

$A \cap A = A$; $A \cap \phi = \phi$; $A \cap S = A$; $S \cap \phi = \phi$

$\phi \cap \phi = \phi$; if $A \supseteq B$ then $A \cap B = B$; $(A \cup B) \cap A = A$; $(A \cap B) \cup A = A$

Disjoint Sets

Two sets A and B having no element in common are disjoint or mutually exclusive $A \cap B = \phi \Leftrightarrow A$ and B are disjoint.



A and B are disjoint

Example:

z^+ and z^- are disjoint

Set of all boys and set of all girls are disjoint

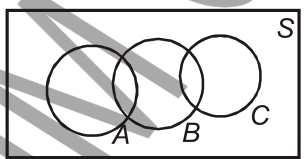
Set of Hindi alphabet and set of English alphabet are disjoint

Set of years of birth of adults and set of years of birth of minors are disjoint

Q and Q' are disjoint

A family of various sets is pairwise disjoint if no two members of this family have a common element.

Note: *If A_1, A_2 .... A_n are pairwise disjoint then $A_1 \cap A_2$ .... $\cap A_n = \phi$, But if $A_1 \cap A_2$ .... $\cap A_n = \phi$, A_1, A_2 .... A_n may not be pairwise disjoint.*



$A \cap B \cap C = \phi$; but A and B, B and C are not disjoint.

So A, B and C are not pairwise disjoint.

ALGEBRA OF SETS

1. **Idempotent Laws :** For any set A, we have
   (a) $A \cup A = A$
   (b) $A \cap A = A$
2. **Identity laws :** For any set A, we have
   (a) $A \cup \phi = A$
   (b) $A \cap \phi = \phi$
   (c) $A \cup U = U$
   (d) $A \cap U = A$
3. **Commutative laws :** For any two sets A and B, we have
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$
4. **Associative laws :** For any three sets A, B and C, we have
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$

5. **Distributive laws :** For any three sets A, B and C, we have
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. **Demorgan's laws :** For any three sets A, B and C, we have
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$
   (c) $A - (B \cup C) = (A - B) \cap (A - C)$
   (d) $A - (B \cap C) = (A - B) \cup (A - C)$
   $(A')' = A$ (for any set A)
   $P(A) \cap P(B) = P(A \cap B)$
   $P(A) \cup P(B) \subseteq P(A \cup B)$

SOME THEOREMS For any sets A, B and C, we have

1. $A - B = A \cap B'$
2. $A \cup B = B \Leftrightarrow A \subseteq B$
3. $A \cap B = A \Leftrightarrow A \subseteq B$
4. $(A - B) \cup B = A \cup B$
5. $A - (B \cup C) = (A - B) \cap (A - C)$
   $A - (B \cap C) = (A - B) \cup (A - C)$

Some more operations on sets.

$A \subseteq A \cup B$; $A \cap B \subseteq A$; $(A - B) \cap B = \phi$

$A \subseteq B \Leftrightarrow B' \subseteq A'$; $A - B = B' - A'$

$(A \cup B) \cap (A \cup B') = A$; $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

$A - (A - B) = A \cap B$; $A - B = B - A \Leftrightarrow A = B$

$A \cup B = A \cap B \Leftrightarrow A = B$; $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

Some Basic Results about Cardinal Numbers

If A, B and C are finite sets and U is finite universal set, then we have

1. $n(A') = n(U) - n(A)$
2. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
3. If $n(A \cup B) = n(A) + n(B) \Leftrightarrow A \cap B = \phi$ *i.e.* A and B are disjoint.
4. $n(A \cap B') = n(A) - n(A \cap B)$
5. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
6. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
7. $n(A - B) = n(A) - n(A \cap B)$
8. $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$
9. $n(A \Delta B) = n(A \cup B) - n(A \cap B)$
10. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

RELATIONS

a R b means 'a is R-related to b' *i.e.* a is related to b under relation R. If $(a, b) \in$ R; (a, b) is called ordered pair in the sense that a and b can't be interchanged as $a \in$ A and $b \in$ B.

Ordered Pair :

It is a pair of objects written in a particular order. Two members are written in a particular order separated by a comma and enclosed in parentheses. Hence in ordered pair (a, b) a is called the first component or the first element or the first co-ordinate and b the second.

Ordered pairs (a, b) and (b, a) are different.

$(a, b) = (c, d)$ iff $a = c$ and $b = d$

i.e. $(1, 3) = (1, 3)$; $(1, 3) \neq (1, 2) \neq (2, 3) \neq (3, 1)$

RELATIONS

For any two non-empty sets A and B, every subset of A × B defines a relation from A to B and every relation from A to B is a subset of A × B.

$a \text{ R } b \subseteq A \times B \quad \forall \text{ R}$

If $(a, b) \in$ R, then a R b is read as 'a is related to b'

If $(a, b) \notin$ R, then $a \not\text{R}\, b$ is read as 'a is not related to b'

Domain and Range of Relation

Domain of R = Dom(R) = Set of first components of all the ordered pairs belonging to R.

Range of R = Set of second components of all the ordered pairs belonging to R.

Co-domain of R = B where R is a relation from A to B

Range of R $\subseteq$ Co-domain of R

$\text{Dom(R)} = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$

$\text{Range of R} = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$

If R = A × B, then Dom(R) = A and Range of R = B

Dom $(\phi) = \phi$; Range of $\phi = \phi$

Representation of a Relation

Let A = {–2, –1, 4} B = {1, 4, 9}

A relation from A to B *i.e.* a R b is defined as a is less than b.

This can be represented in the following ways.

1. **Roster form:**

   R = {(–2, 1), (–2, 4), (–2, 9), (–1, 1), (–1, 4), (–1, 9), (4, 9)}

2. **Set builder notation:**

   R = $\{(a, b): a \in A \text{ and } b \in B, a \text{ is less than } b\}$

Inverse Relation

Let R $\subseteq$ A × B be a relation from A to B.

The inverse relation of R (denoted by R^{-1}) is a relation from B to A defined as $R^{-1} = \{(b, a) : (a, b) \in R\}$

If $(a, b) \in R$, then $(b, a) \in R^{-1}$, $\forall\, a \in A, b \in B$.

domain of R^{-1} = Range of R

Range of R^{-1} = domain of R

$(R^{-1})^{-1} = R$

Identity Relation

The identity relation on a set A is the set of ordered pairs belonging to A × A and is denoted by I_A.

$I_A = \{(a, a) : a \in A\}$

i.e. every element of A is related to only itself.

R is an identity relation if $(a, b) \in R$ iff $a = b$, $a \in A$, $b \in A$.

$\mathbf{I_A^{-1} = I_A}$

Domain of I_A = *Range of* $I_A = A$

Illustration 20 :

'is equal to' is an identity relation on set of Natural number (N)

i.e. $\{(1, 1), (2, 2), (3, 3)........\} = I_N$

Universal Relation

If A be a set and R is the set A × A, then R is called universal relation in A.

If $R = A \times A$, *then R is universal relation in A.*

Void Relation

ϕ is called the empty or void relation if $\phi \subset A \times A$

Types of Relations on a Set

If A is a non-empty set, then a relation R on A is said to be

1. Reflexive :

If $(a, a) \in R, \forall a \in A$.

i.e. $a R a, \forall a \in A$

"is equal to", "is a friend of", "is parallel to", are some of reflexive relations.

2. Symmetric :

If $a R b \Rightarrow b R a, \forall a, b \in A$

i.e. if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

"is a friend of", "is parallel to", "is equal to", are some of symmetric relations.

3. Anti-Symmetric :

If $a R b$ and $b R a \Rightarrow a = b, \forall a, b \in A$ (If $R \cap R^{-1}$ = Identity, then R is anti-symmetric)

"is divisible by" is an anti symmetric relation.

4. Transitive :

If $a R b$ and $b R c \Rightarrow a R c, \forall a, b, c \in A$

i.e. If $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$

"is parallel to", "is equal to", "is congruent to" are some of the transitive relation.

Equivalence Relation

A relation R on a non-empty set A is called an equivalence relation if and only if it is Reflexive, Symmetric as well as Transitive.

"is parallel to", "is equal to", "is congruent to" "Identity relation" are some of the equivalence relations.

Every identity relation is an equivalence relation but every equivalence relation need not to be identity relation.

FUNCTION

Let A and B are two non empty sets. A function f from set A to set B is a rule which associated each element of A to a unique element of B, denoted by $f : A \to B$

set A is called domain of function 'f'

set B is called co-domain of function 'f'

If element x of A corresponds to $y(\in B)$ under the function f, then we say that y is the image of x and write $f(x) = y$.

Domain of the Function

Domain of the function is set of all those real numbers (x) for which $f(x)$ exists or $f(x)$ is meaningful. $f(x) \neq \infty$ or any imaginary no.)

Range of Function

Set of all the images of elements in domain is called the range.

Range = $\{f(x) : x \in \text{domain}\}$

Algebraic Operation on Functions

1. Given functions f and g, their sum $f + g$, difference $f - g$, and fg are defined on dom $f \cap$ dom g as: $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$ and $(fg)(x) = f(x)\, g(x)$. Moreover f/g is defined on dom $f \cap \{x \in \text{dom } g : g(x) \neq 0\}$ by $(f/g)(x) = f(x)/g(x)$.
2. If k is any real number and f is a function then kf is defined on the domain of f by $(k f)(x) = k f(x)$.

   We have the following formulae for domains of functions

   (i) dom $(f \pm g)$ = dom $f \cap$ dom g

   (ii) dom $(f g)$ = dom $f \cap$ dom g

   (iii) dom (f/g) = dom $f \cap \{x \in \text{dom } g : g(x) \neq 0\}$

   (iv) dom $\sqrt{f} = \{x \in \text{dom } f;\ f(x) \geq 0\}$

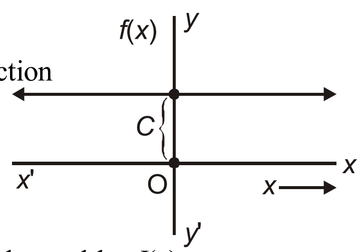
SOME STANDARD FUNCTIONS AND THEIR GRAPHS

Constant Function

A function denoted by $f(x) = C$ (where $C \in R$) is known as constant function

Domain = R

Range = C



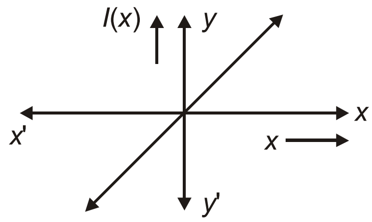
Identity Function [$I(x)$] :

A function which is associated to itself is known as identity function and denoted by $I(x) = x$

Since x can take any value so domain of this function is R, corresponding value of $I(x)$ is also R, so range is R

Domain = R

Range = R

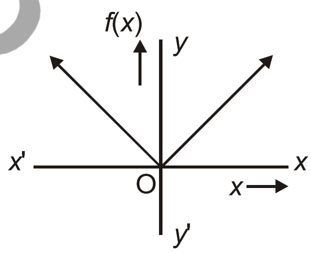


Modulus Function :

This is also known as absolute value function and denoted by

$f(x) = |x|$

i.e. $f(x) = \begin{cases} x\ , & x \geq 0 \\ -x\ , & x < 0 \end{cases}$



Domain of this function is set of all real numbers because $f(x)$ exists for all $x \in R$ but $|x| \geq 0$ so range is all non-negative real numbers.

Domain = R

Range = $[0, \infty]$

or $R^+ \cup \{0\}$

Properties of modulus function :

(a) $| x |^n = | x^n |$

(b) $| x^n | = x^n$, where n is even and $n \in z$

(c) $| x\,y | = | x |\,| y |$

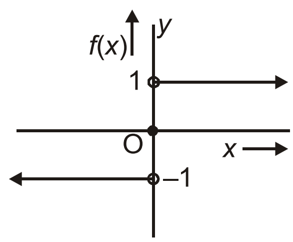
(d) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, $(y \neq 0)$

(e) $|\,| x | - | y |\,| \leq | x + y | \leq | x | + | y |$

Signum Function

The function $f(x)$, defined as $f(x) = \begin{cases} \dfrac{|x|}{x} & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$

is called signum function. This signum function may also defined as



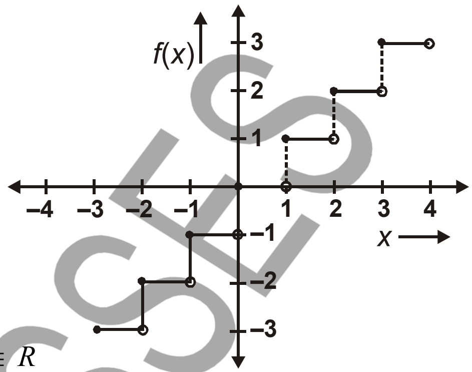
$f(x) = \begin{cases} 1 & ; \ x > 0 \\ 0 & ; \ x = 0 \\ -1 & ; \ x < 0 \end{cases}$

Domain = R

Range = $\{-1, 0, 1\}$

Greatest Integer Function

This function is also known as step function or floor function denoted by $f(x) = [x]$. By $[x]$ we mean greatest integer less then or equal to x. If n is an integer and x is any real number between n and $n + 1$



i.e. $n \le x < n + 1$, then $[x] = n$

Thus $[3.4] = 3$, $[3.99] = 3$

$[-4.99] = -5$, $[-4.001] = -5$

$[0.3] = 0$, $[-0.2] = -1$

Domain of $[x]$ is set of all real numbers because $[x]$ exist $\forall x \in R$

But $[x]$ is always integral number so range is set of all integers Z.

Some Properties of [x] :

(a) $[x + k] = [x] + k$, if $k \in Z$

(b) $[-x] = -[x] - 1$

(c) $[x] + [-x] = 0$, $x \in Z$

(d) $[x] + [-x] = -1$, $x \notin Z$

(e) $[x] - [-x] = 2x$, $x \in Z$

(f) $[x] - [-x] = 2[x] + 1$, $x \notin Z$

(g) $x - 1 < [x] \le x$

(h) $\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\left[x+\frac{3}{n}\right]+\ldots\ldots\ldots\ldots\left[x+\frac{n-1}{n}\right] = [nx]$

(i) $[x + y] \ge [x] + [y]$

(j) $\left[\frac{(x)}{n}\right] = \left[\frac{x}{n}\right], n \in N$

(k) $[x] \ge n \Rightarrow x \ge n$, $n \in z$

(l) $[x] > n \Rightarrow x \ge n + 1$, $n \in z$

(m) $[x] \le n \Rightarrow x < n + 1$, $n \in z$

(n) $[x] < n \Rightarrow x < n$, $n \in z$

Fractional Part Function :

Function denoted by $f(x) = \{x\}$, known as fractional part function.

Also defined as $f(x) = x - [x]$

If $x \in Z$, then $f(x) = 0$ [*i.e.* $f(2) = 2 - [2] = 0$]

If $x \notin Z$, then $f(x)$ lies between 0 to 1.

i.e. $x \notin Z$, $0 < f(x) < 1$ [*i.e.* $f(3.4) = 3.4 - [3.4] = 3.4 - 3 = 0.4$]

Note : *Fractional part function is a periodic function having period '1'.*

Domain = R

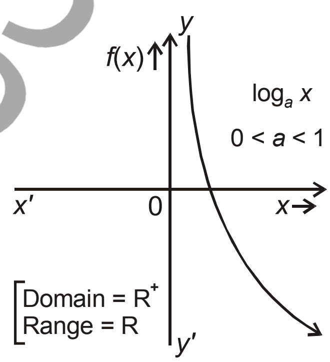
Range [0, 1)

LOGARITHMIC FUNCTION

If $f : R^+ \to R$, $f(x) = \log_a x$, then $f(x)$ is known as logarithmic function

Here $f(x)$ exist if $x > 0$ and $0 < a < 1$ or $a > 1$ $(a \neq 1)$

Properties of logarithmic function



(i) $\log_a m.n = \log m + \log n$

(ii) $\log_a \frac{m}{n} = \log m - \log n$

(iii) $\log_a m^n = n\log_a m$

(iv) $\log_{a^q} b^p = \frac{p}{q}\log_a^{b}$

(v) $\log_a^{b} = \frac{\log_x b}{\log_x a} = \log_x b.\log_a x$

(vi) $\log_a^b.\log_b^a = 1$

(vii) If $\log_a f(x) = y \quad \Rightarrow \quad f(x) = (a)^y$

(viii) If $\log_a f(x) \geq \log_a g(x) \quad \Rightarrow \quad \begin{cases} f(x) \geq g(x) \text{ if } a > 1 \\ f(x) \leq g(x) \text{ if } 0 < a < 1 \end{cases}$

(ix) If $\log_a f(x) \geq y \quad \Rightarrow \quad \begin{cases} f(x) \geq (a)^y \; a > 1 \\ f(x) \leq (a)^y \; if \; 0 < a < 1 \end{cases}$

(x) If $\log_a f(x) \leq y \quad \Rightarrow \quad \begin{cases} f(x) \leq (a)^y \quad a > 1 \\ f(x) \geq (a)^y \; 0 < a < 1 \end{cases}$

Exponential Function

$f(x) = a^x$ is known as exponential function $(a > 0)$

KINDS OF FUNCTION

Polynomial Function

The function $f(x) = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n$ where $a_0, a_1, a_2, \dots\dots a_n \in R$ and $n \in N$ is called a polynomial function of degree n.

Rational Function

A function defined by the quotient of two polynomial function is called rational function for

Example :

$$\frac{x^2+1}{x^3+x+1} \text{ is a rational function.}$$

Irrational Function

A function involving one or more radicals of polynomial is called a irrational function

Example :

$$x^{\frac{3}{2}}+\sqrt{x}+x^2, \quad \frac{x^2+2x+3}{x+\sqrt[3]{x}+5} \text{ etc.}$$

Algebraic Function

An algebraic function is one which consist of a finite number of terms involving power and roots of the variable x and simple operation, addition, subtraction, multiplication and division *i.e.* all rational, and irrational functions are algebraic functions.

Transcendental Function

All function which are not algebric are called transcendental function.

Example :

(a) All trigometric function *i.e.* $\sin x$, $\cos x$ etc.

(b) All exponential function, e^x, $\log x$, a^x etc.

(c) Inverse trigonometric function $\sin^{-1} x$, $\cos^{-1} x$, etc.

Note : *A transcendental function is not expressed in a finite number of algebraic terms.*

Explicit Function

A function in which dependent variable (y) is expressed directly in terms of independent variable (say x)

i.e. $y = x^3 + x^2 + 1$, $y = \dfrac{x^2+3x+5}{x+2}$, etc.

Implicit Function

A function in which we can't express dependent variable in terms of independent variable.

Example:

$x^3 + y^3 + 3xy = 0$, note that we can't write y or x in terms of x, or y separately.

Even or Odd Function

(a) Even function : If $f(-x) = f(x)$ then $f(x)$ is said to be even function.

Example : $f(x) = \cos x$ is a even function [$\because f(-x) = \cos(-x) = \cos x = f(x)$]

(b) Odd function : If $f(-x) = -f(x)$ then $f(x)$ is said to odd function.

Note : *(a) Even function is symmetrical about y-axis while odd function is symmetrical about origin (i.e. in opposite quadrant)*

(b) Addition and subtraction of two even function is always even function.

(c) Sum of even and odd function is neither even nor odd function.

(d) Any function 'f' can be represented as the sum of an even and an odd function.

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(n) - f(-x)]$$

where $\frac{1}{2}[f(x) + f(-x)]$ *is an even and* $\frac{1}{2}[f(x) - f(-x)]$ *is an odd function*

(e) $f(x) = 0$ *is the only function which is both odd and even.*

Periodic Function

A function 'f' defined on its domain is said to be periodic function if their exist a positive number T such that $f(x + T) = f(x)\ \forall\ x \in$ D. Also both $x + T$ and $x - T$ should belong to D.

The least value of T, it exists is called, the period of the function.

Some Standard Functions and their Period

Function	**Period**
$\sin x$	2π
$\cos x$	2π
$\tan x$	π
$\{x\}$	1

Some Special Point about Periodic Function

If period of $f(x)$ is 'T' then

(a) (i) Period of $|f(x)|$ is $\frac{T}{2}$.

(ii) Period of $[f(x)]^n$ is $\frac{T}{2}$, if n is even number ($n \in$ N)

(iii) Period of $[f(x)]^n$ is T, if n is odd number ($n \in$ N)

(iv) Period of $f(ax)$ and $f(ax + b)$ is $\frac{T}{|a|}$.

(v) Period of $f\left(\frac{x}{a}\right)$ is $|a|$.T.

(b) If Period of $f(x)$ and $g(x)$ are same say 'T' then period of $f(x) \pm g(x)$ is given by

(i) $\frac{T}{2}$ (if $f(x)$ and $g(x)$ both are even).

(ii) T (if $f(x)$ is any function except even).

(c) If period $f(x)$ is T_1 and $g(x)$ is T_2. Then period of $f(x) \pm g(x)$ is given by L.C.M. of T_1 and T_2 (same for $\frac{f(x)}{g(x)}$)

Note : *(i) LCM of* $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} = \frac{\text{LCM of } a,c,e}{\text{HCF of } b,d,f}$.

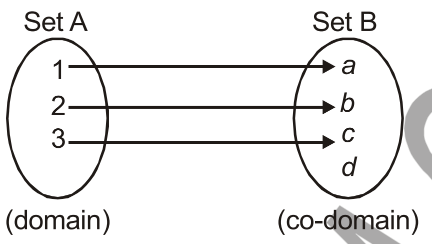
(ii) Sin $\sqrt{x}$ *and sin* x^2 *is not a periodic function because these can't be written in the form of [f(x + T) = f(x)]*

(iii) L.C.M. of rational with irrational is not possible, e.g., L.C.M. of (π, 2, 2π) is not possible as π, 2π ∈ irrational and 2 ∈ rational

TYPES OF MAPPINGS OR FUNCTIONS

One-one Function or Injective Function :

A function is said to be one-one function if different element in a domain have different images in co-domain.



if $f(x_1) = f(x_2)$ then $x_1 = x_2$
$f(x)$ is one - one function

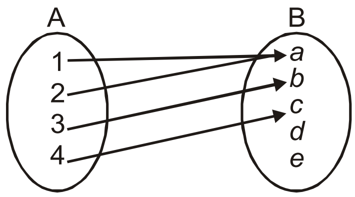
Note : *(i)* ***Example of one-one function :*** *Linear polynomial function (ax + b), x,* e^x*, log x, are always one-one functions.*

(ii) If $\frac{dy}{dx} > 0$ *or* $\frac{dy}{dx} < 0$ $\forall x \in$ domain*, then y = f(x) is said to be one-one function.*

Number of one-one function : If A and B are finite sets having m and n elements respectively, then number of one-one function from A to B = ^nP_m*, if* $n \geq m$ *= 0, if* $n < m$.

Many–one Function

A function $f : A \to B$ is said to be many one if more than one element in set A have same image in Set B.

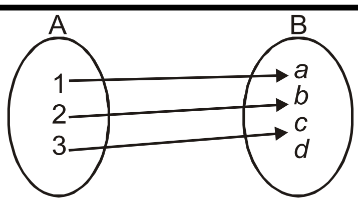


Note : (i) All even function, modulus function, periodic function are always many-one function.

(ii) Square function, Trigonometric function are also many–one function in their domain.

Into Function

A function $f: A \to B$ is said to be into function if there exist at least one element in set B having no any pre-image in set B.



In fig set B (co-domain) there is no pre-image, for element d, in set A, so function is into function.

Onto Function

f : A → B, said to be onto function if every element in set B has a pre image in set A.

Range of f = co-domain of f.

Example of Onto function :

log x, linear polynomials, are always onto function.

Possible mappings are

(i) One-one and onto (bijective function)

(ii) Many one and onto

(iii) One-one and into

(iv) Many one-into

Example :

If $f : R \to R$, $f(x) = x^2 + 3x + 2$ then $f(x)$ is many one function.

because $f(x) = x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$

$$f(-2) = \left(-2 + \frac{3}{2}\right)^2 - \frac{1}{4} = 0$$

$$f(-1) = \left(-1 + \frac{3}{2}\right)^2 - \frac{1}{4} = 0$$

So image of –2 and –1 are same

$\therefore$ $f(x)$ is many one.

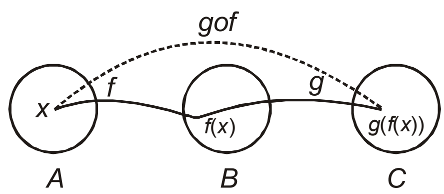
Example :

$f(x) = 2x + \sin x$, is one-one because $f'(x) = 2 + \cos x$, minimum value of $\cos x$ is –1.

$\therefore$ $f'(x) > 0$ for all $x \in$ Domain = R

Composition of Function

Let $f : A \to B$ and $g : B \to C$ then the composition of g and f is denoted by gof and is defined as $gof : A \to C$ given by $gof(x) = g(f(x))$



Similarly fog is defined. Note that, gof is defined only if Range $f \subseteq$ dom g and fog is defined only if Range $\subseteq$ dom f. dom fog = $\{x \in \text{dom } g : g(x) \in \text{dom } f\}$

Illustration 2 :

Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Since dom $g = [0, \infty)$, dom $f = R$

we have $fog\ (x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 3 = x + 3$

So dom $fog = \{x \in [0, \infty) : g(x) \in R\} = [0, \infty)$

Let us now find gof, we have $(gof)\ (x) = g(f(x)) = g(x^2 + 3) = \sqrt{x^2 + 3}$,

then dom $gof = \{x \in R : f(x) \in [0, \infty)\} = R$.

Inverse Function

Two functions f and g are inverse of each other if $f\ (g(x)) = x$ for $x \in$ dom g and $g(f(x)) = x$ for $x \in$ dom f, *i.e.*, $gof = I_{dom} f$ and $fog = I_{dom}\ g$ where $I_{dom} f$ is identity function on dom f and $I_{dom}\ g$ is identity function on dom g. We denote g by f^{-1} or f by g^{-1}. To find the inverse of f, write down the equation $y = f(x)$ and then solve x as a function of y. The resulting equation is $x = f^{-1}\ (y)$.